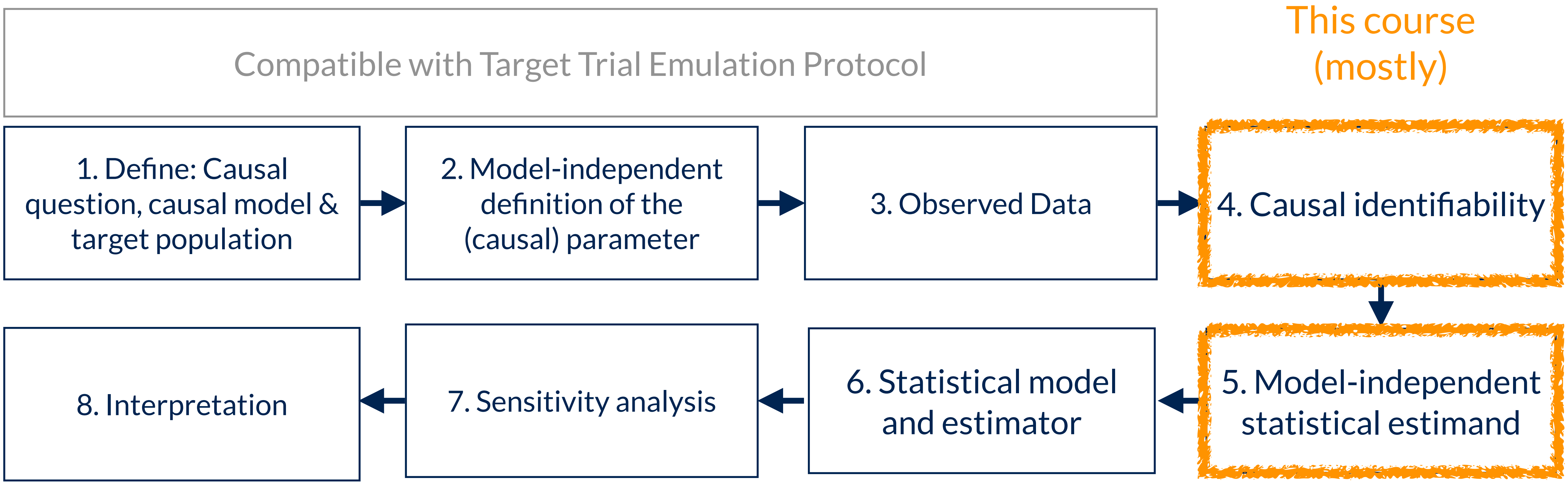
THE UNIVERSITY
of EDINBURGH

Methods for Causal Inference
Lecture 2: Basics of probability

Ava Khamseh

School of Informatics
2025-2026

The Causal Roadmap



Compatible with Target Trial Emulation Protocol
This course (mostly)
1. Define: Causal question, causal model & target population
2. Model-independent definition of the (causal) parameter
3. Observed Data
4. Causal identifiability
5. Model-independent statistical estimand
6. Statistical model and estimator
7. Sensitivity analysis
8. Interpretation

A causal roadmap for generating high-quality real-world evidence
https://doi.org/10.1017/cts.2023.635

The Causal Roadmap (summary of steps 1-3)

Step 1: Specifying the **causal question(s)** of interest, the study's target population, the exposure(s), outcomes(s), time period and context, as well as the **causal model** (developed with input from subject experts) to describe relationships amongst the variables.

Step 2: Define of the causal estimand, which is a mathematical quantity representing the answer to the causal question(s)

Step 3: Observed data, assesses questions such as what is the baseline time zero, are the required variables in the causal model of step 1 measured, or measured differently in various data sources. If these essential ingredients are not recorded in the observed data, the causal question in the first step may have to be adapted accordingly.

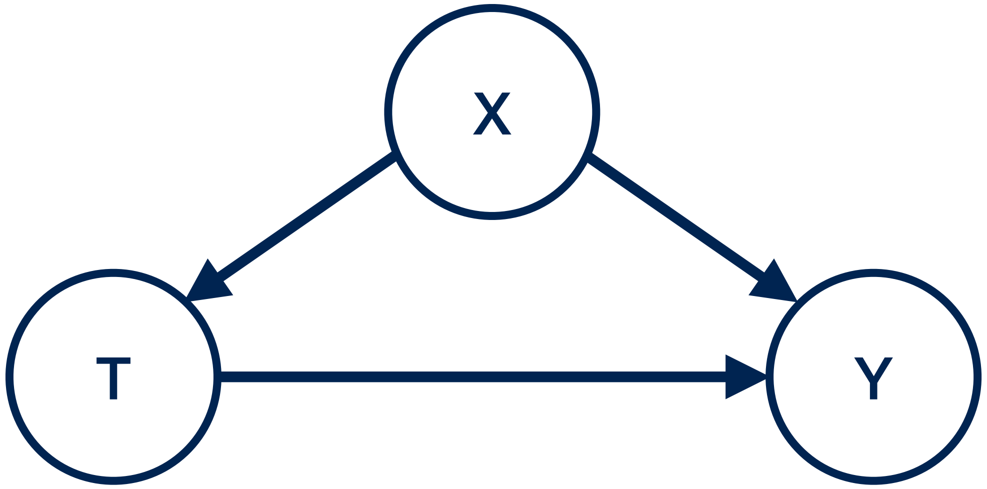
Causal theory and data

Disclaimer: In this course our focus is on steps 4-5. We then use simple models to exemplify 6-7 (taking model assumptions as ‘true’),
i.e., we do not discuss valid **statistical inference**.
For causal/statistical inference please refer to the course:
Targeted Causal Learning (code: MATH11238).

Two main Frameworks for causal identifiablity

- Potential outcomes framework (Neyman-Rubin):

- Requires a given treatment-outcome pair (known directionality)
- For causal estimation
- More familiar to biomedical researchers (this is changing …)



- Structural causal models (Pearl):

- Causal graphs
- Structural equations $x = f_x(\epsilon_x),\ t = f_t(x, \epsilon_t),\ y = f_y(x, t, \epsilon_y)$
- Algorithmic
- For causal estimation and discovery

Assumption: Independent noise terms: $\epsilon_x \perp\!\!\!\perp \epsilon_t \perp\!\!\!\perp \epsilon_y$

Overview of the course

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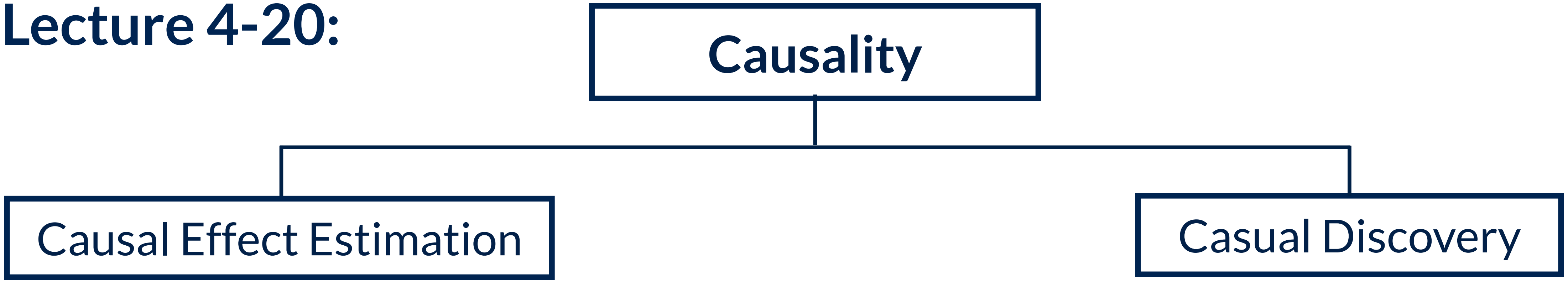

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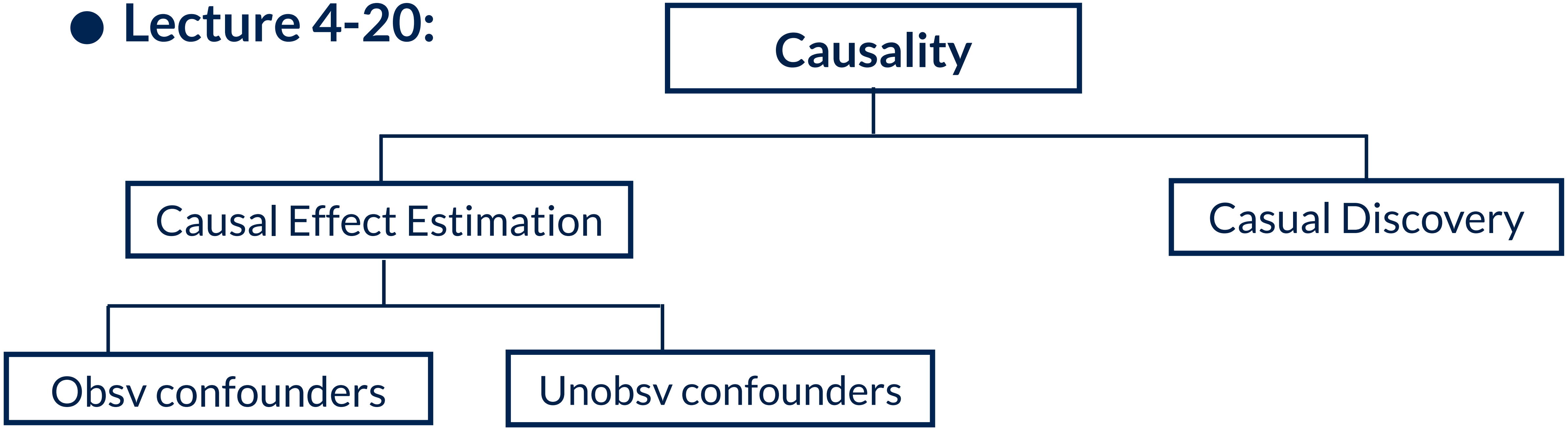

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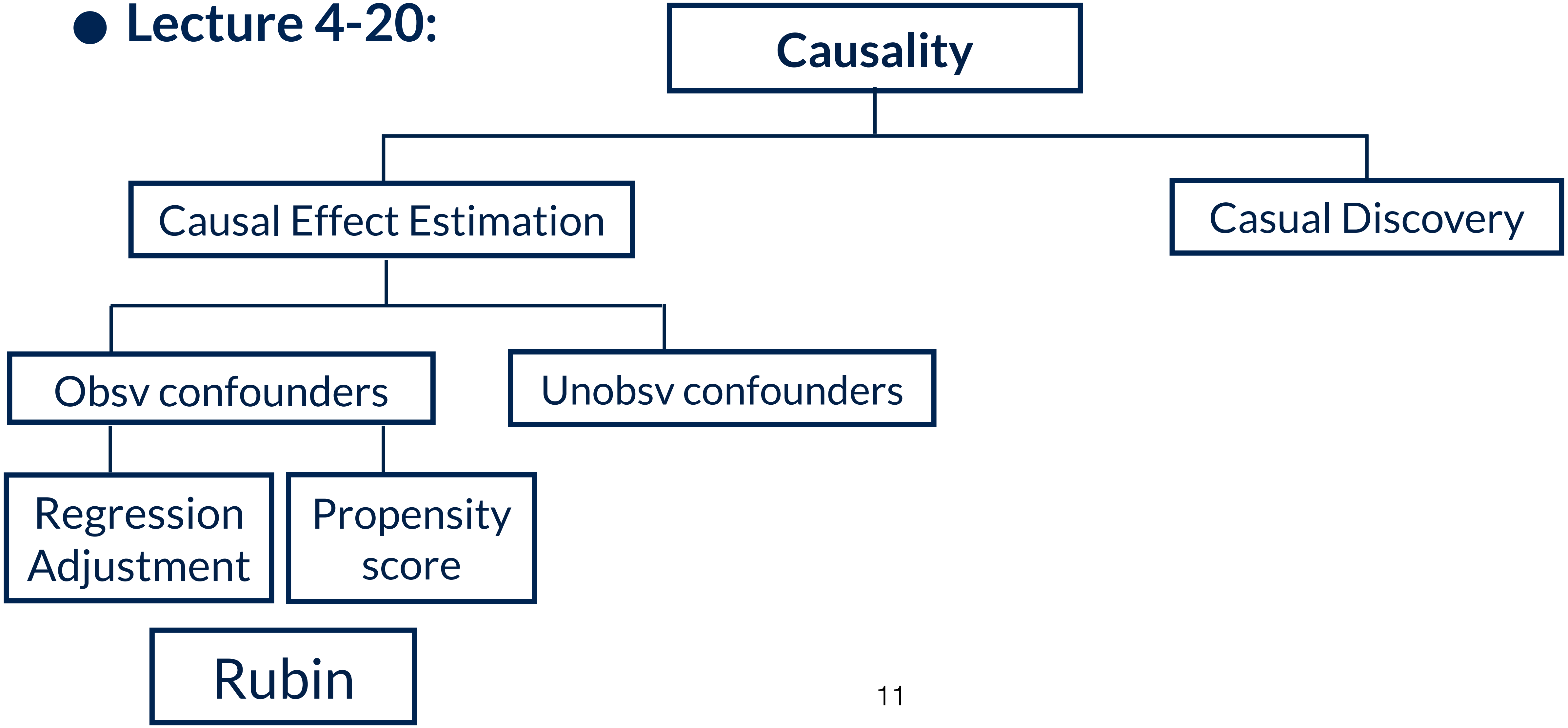

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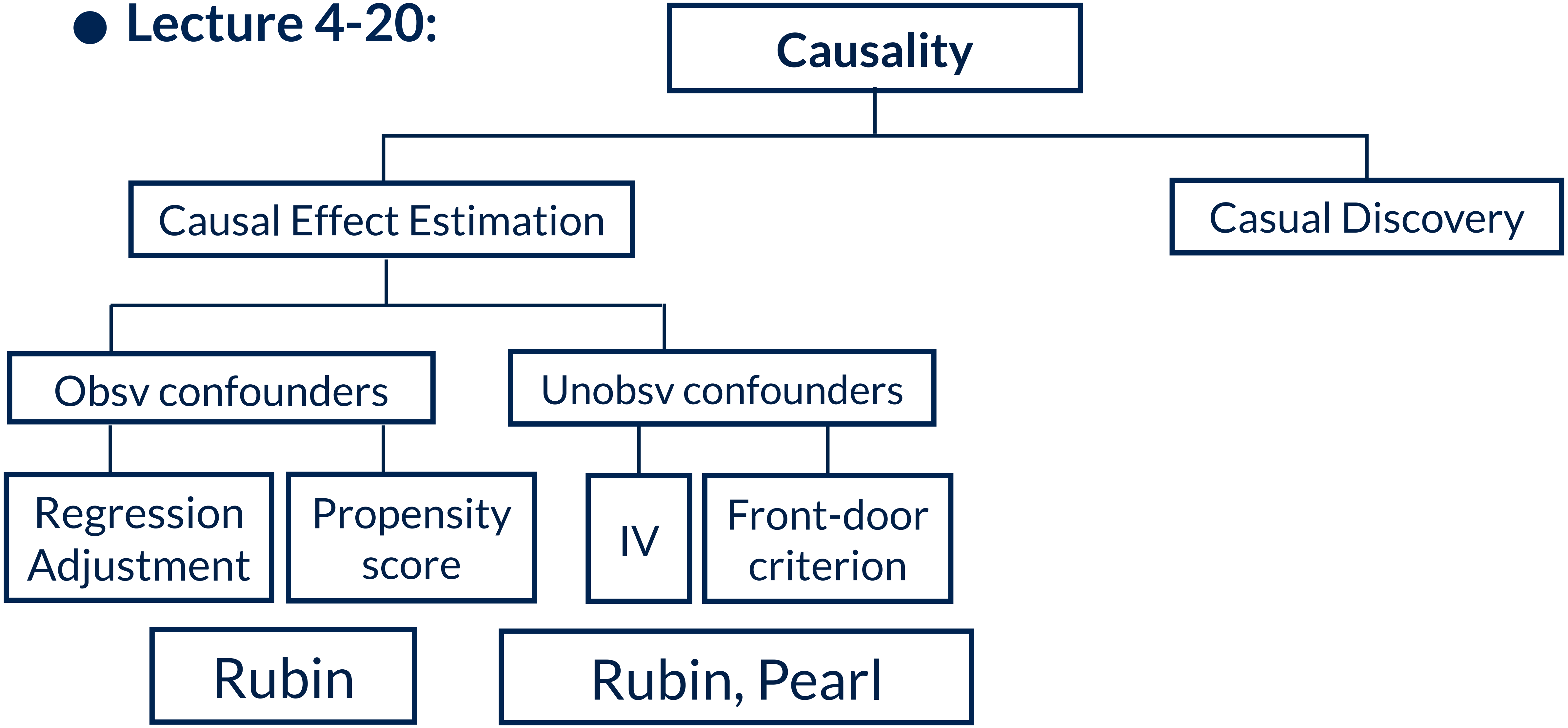

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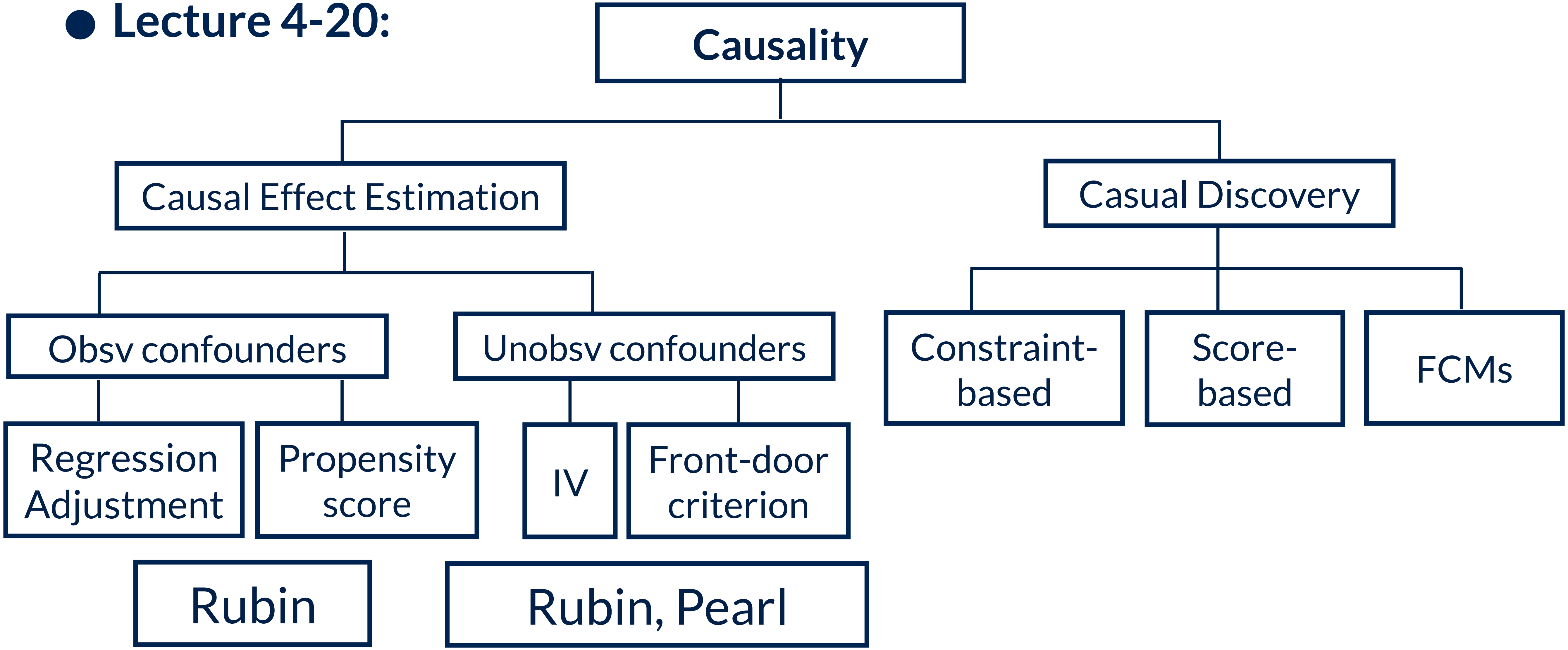

Causality

Causal Effect Estimation

Casual Discovery

Obsv confounders

Unobsv confounders

Constraint-based

Score-based

FCMs

Regression Adjustment

Propensity score

IV

Front-door criterion

Rubin

Rubin, Pearl

Lecture 2: Recap of probability theory

Causal theory and data

Defining causation:
A variable X is a cause of a variable Y if Y in any way relies on X for its value. (Intuitively: X is a cause of Y if Y listens to X and decides its value in response to what it hears)

Pre-requisites: Elementary concepts from probability theory, statistics, graph theory

Basics of probability

Most causal statements are uncertain: “drinking causes liver disease”, does not mean every person who consumes alcohol is certain to have liver disease

→ Need language and laws of probability.

Causal Inference in Statistics, Pearl (2016)

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(Random) variables: Any property or descriptor that can take multiple values, e.g., age ($x=40$), sex ($x'=F$), family history of disease ($x''=0$), … .

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Example: Individual > 40 and recovered from covid $y=0$, event is ($x > 40, y=0$). So variables are ‘age’ and ‘recovery status’ with values > 40 and 0.

Can ask what is the probability of an event, e.g., what is $P(x > 40, y=0)$?

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Events: An event is any assignment of a **value or set of values** to a variable or set of variables.

Discrete (binary/categorical): Are being treated or not, have a disease or not, …

Continuous (can take infinite set of values): age, weight, …

Drug (yes/no) vs dose of drug (categorical). Sun intake (time is continuous)

Basics of probability

For probabilistic modelling (of a random experiment) we need to:

- Describe possible outcomes: **sample space**
- **Event:** A subset of sample space
- Describe beliefs about likelihood of these events: **probability law**

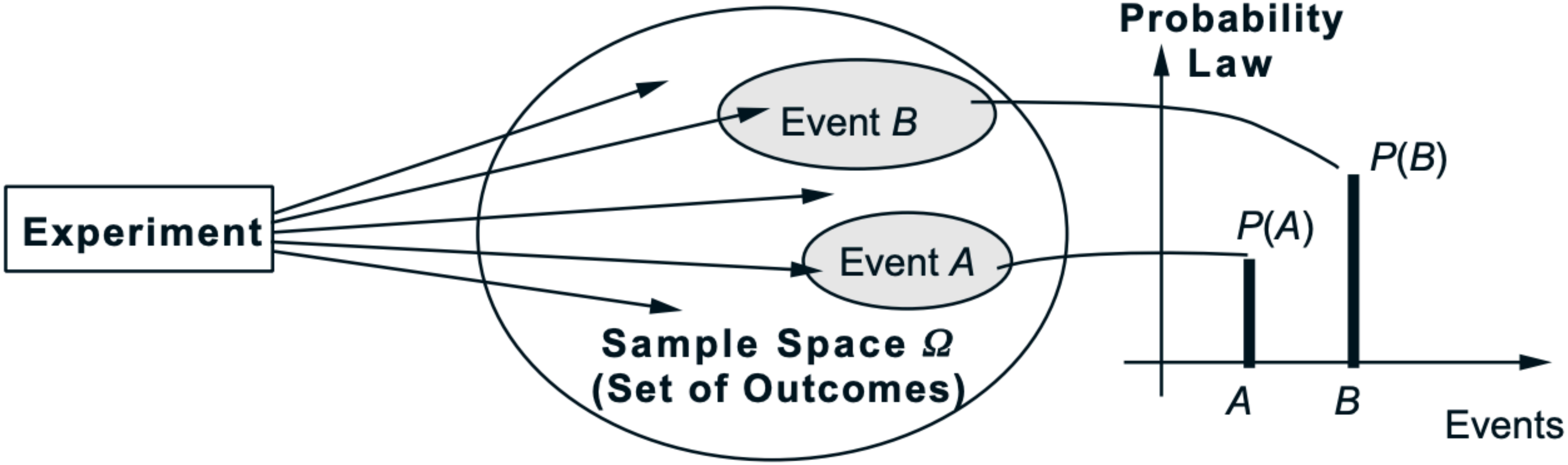
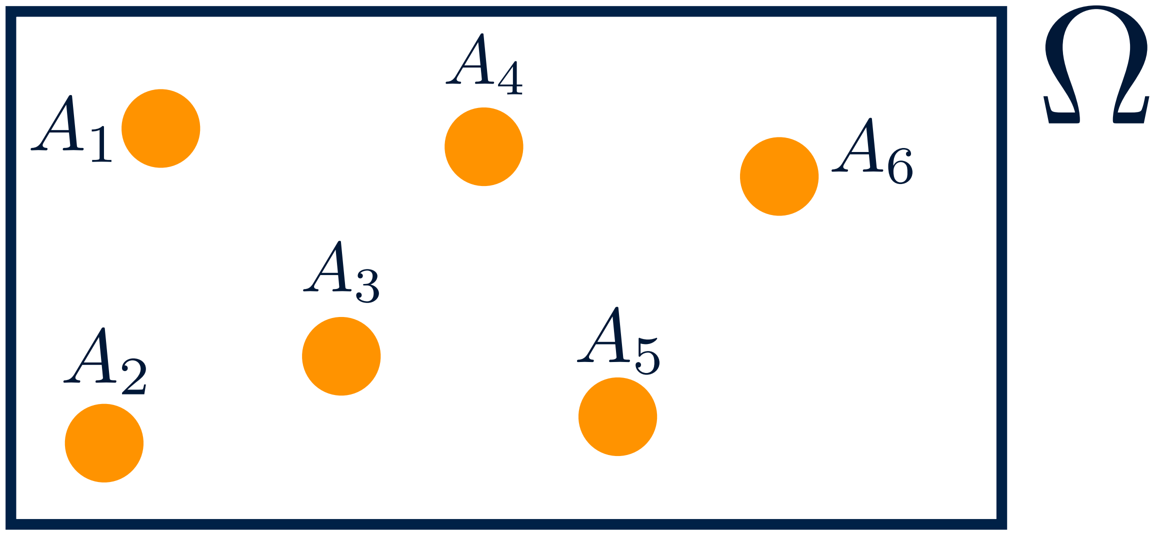


Figure from Tsitsiklis MIT Lecture Notes: Introduction to probability

Sample space

The sample space is the set of all possible outcomes of the experiment:

e.g. Rolling a dice



Outcomes must be:

- **Mutually Exclusive**: If I tell you, after the experiment, that A_1 happened, then it should not be possible that A_6 also happened.
- **Collectively Exhaustive**: Collectively, all the outcomes in Ω exhaust all possibilities.

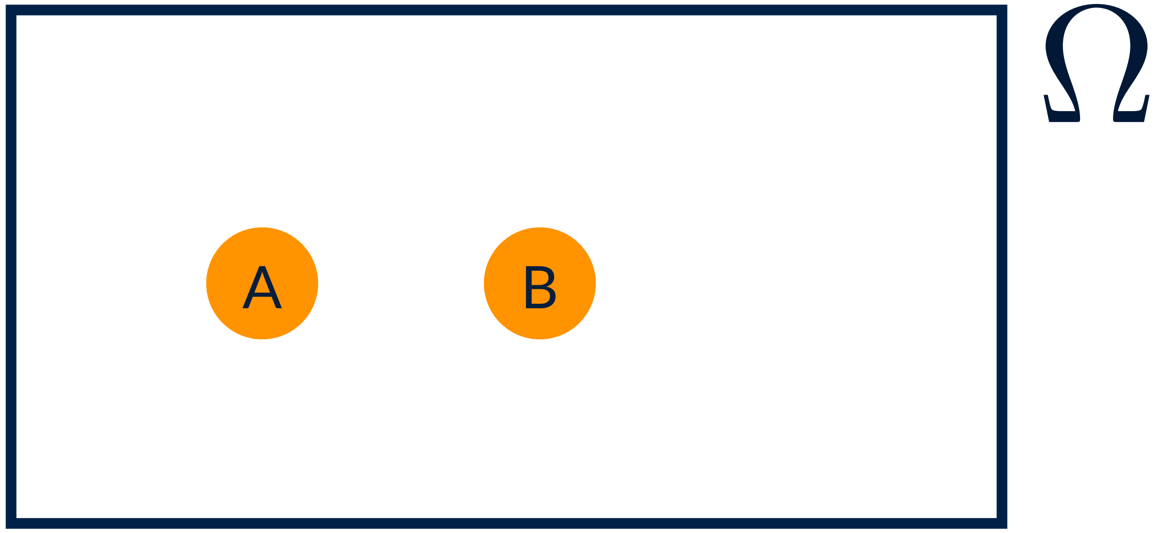
Tsitsiklis: Introduction to probability course, MIT

Probability Axioms

- Non-negativity: $P(A) \geq 0$
- Normalisation: $P(\Omega) = 1$

- For any two **mutually exclusive events** (i.e. A and B cannot co-occur) we have:

$$P(A \text{ or } B) = P(A) + P(B)$$



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As a consequence, take any two events A and B (they may overlap!), then:

$$P(A) = P(A \text{ and } B) + P(A \text{ and `not } B\text{'})$$

Mutually exclusive: If A is true, either “A and B” or “A and not B” must be true.

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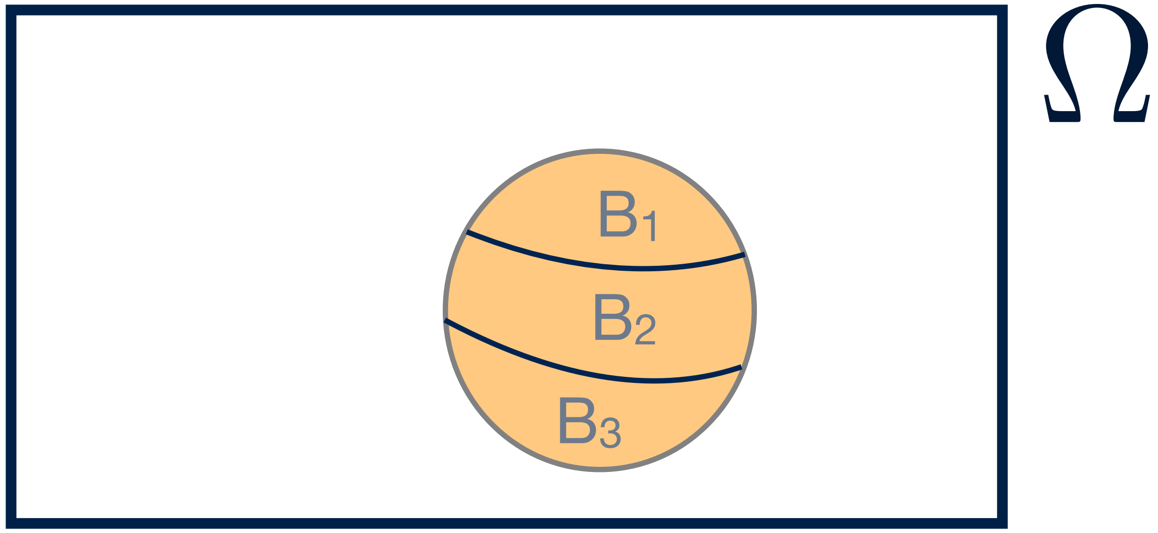

Corollary: B_1, B_2, B_3, are exclusive, and together form all of B. Then,

$$P(A \text{ and } B) = P(A \text{ and } B_1) + P(A \text{ and } B_2) + P(A \text{ and } B_3)$$

Generalise for (exhaustive, mutually exclusive) **partitions** of B:

$$P(A \text{ and } B) = \sum_{i=1}^{n} P(A \text{ and } B_i) \quad \text{where } B_i \cap B_j = \emptyset, \bigcup_{i=1}^{n} B_i = B$$

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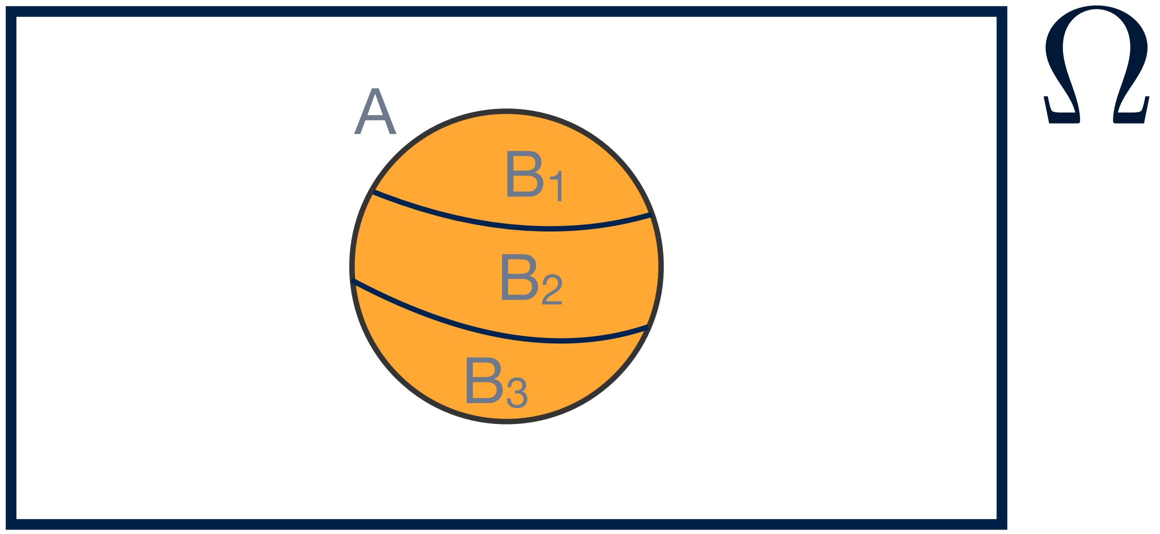


Corollary: Let B$_i$, i=1, …,n be mutually exclusive and exhaustive partitions of B, and let A=B (complete overlap). Then,

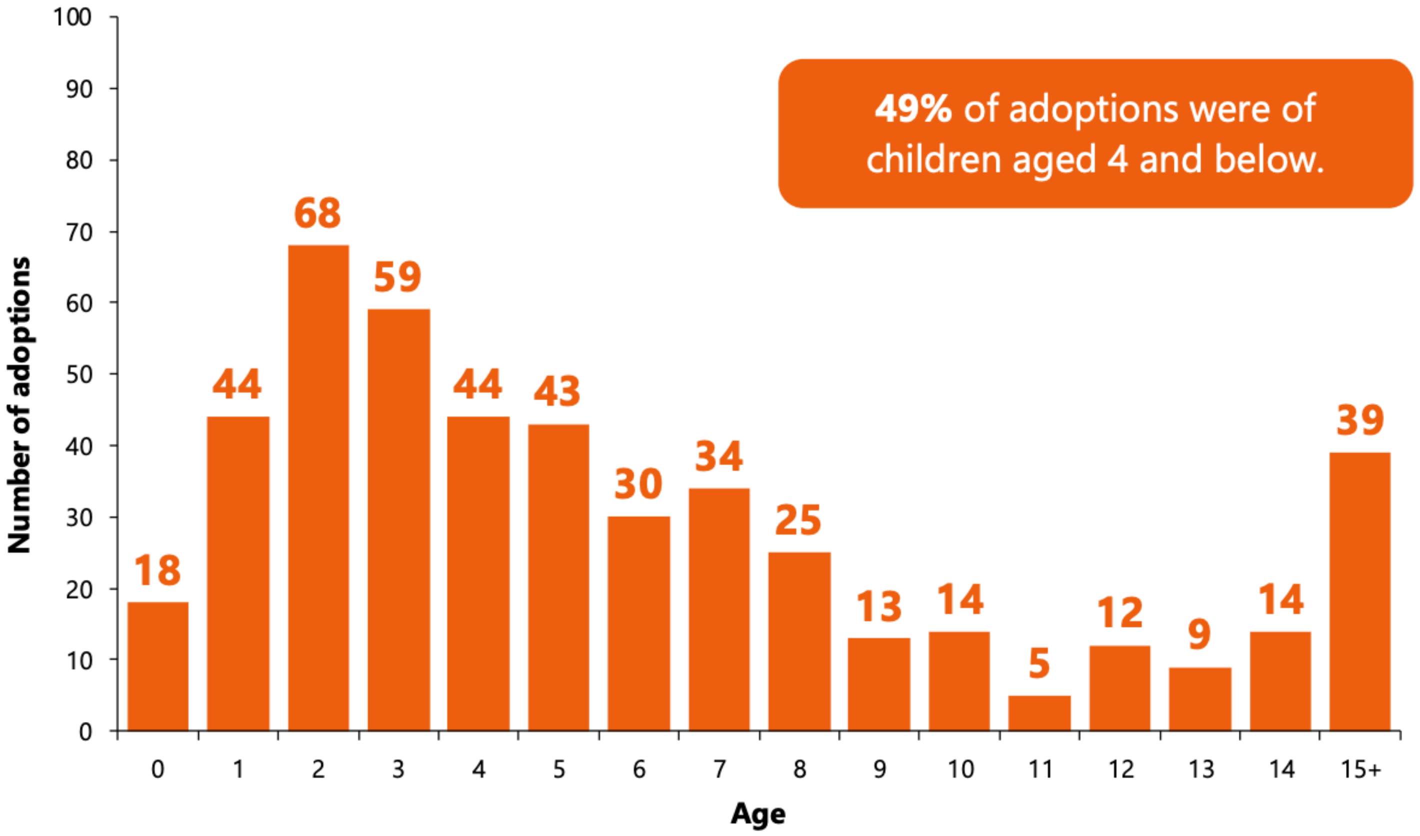
$$P(A) = P(A \text{ and } A) = P(A \text{ and } B) = \sum_{i=1}^{n} P(A \text{ and } B_i) \quad \text{where } B_i \cap B_j = \emptyset, \bigcup_{i=1}^{n} B_i = B$$

Total Law of probability. See later: “marginalisation”

Intervals

P(age > 4) = 1 - P(age <= 4) = 1- 0.49 = 0.51

Figure 7.2: Age at adoption, Scotland, 2018



49% of adoptions were of children aged 4 and below.

Total = 471

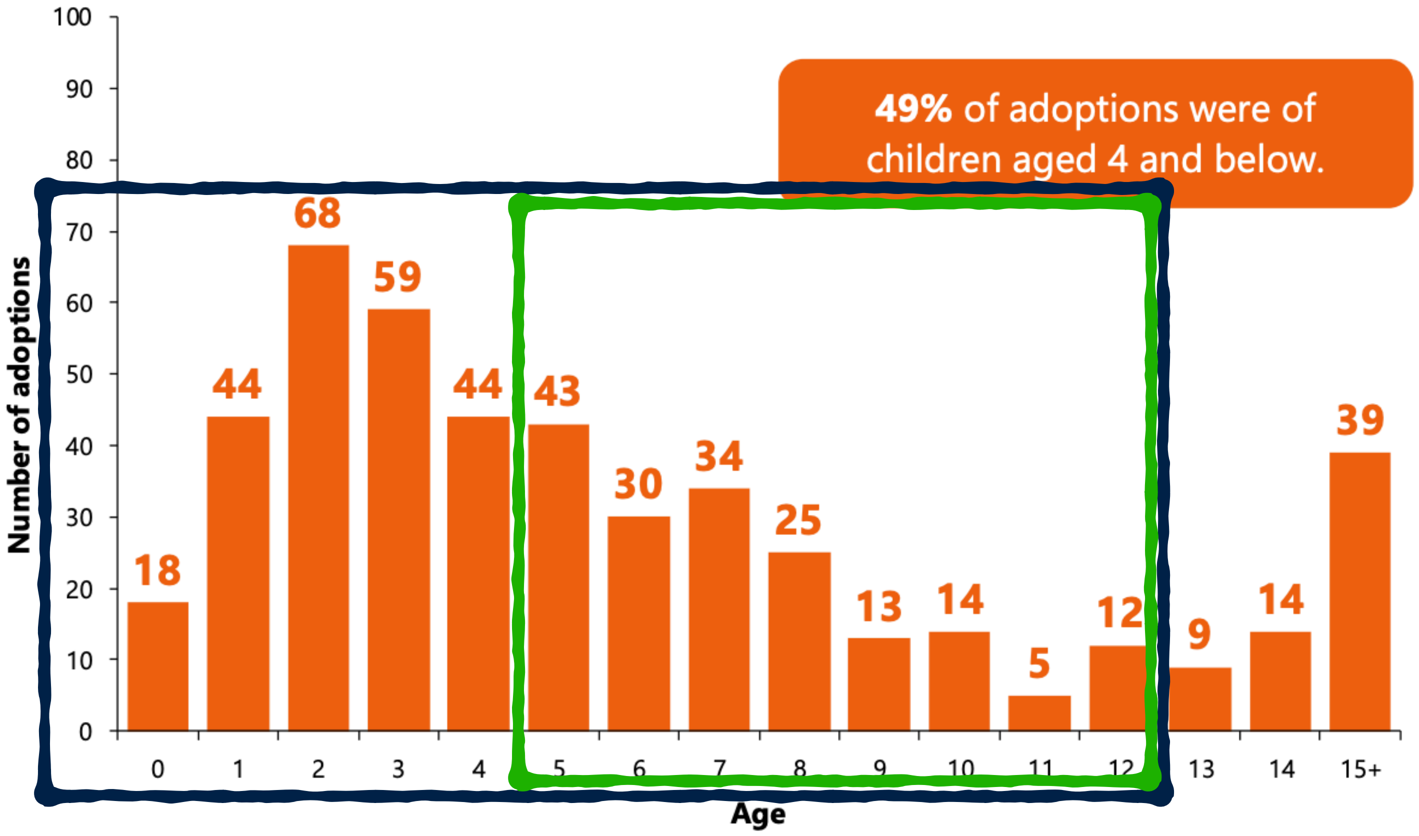
National Records of Scotland 2018

Intervals

P(age > 4) = 1 - P(age <= 4) = 1- 0.49 = 0.51

P(4 < age <= 12) = (43+30+34+25+13+14+5+12) / 471 = 0.37

Figure 7.2: Age at adoption, Scotland, 2018



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Conditional Probability

The probability that event A occurs, given that we know some other event B has occurred. (Think of filtering the data based on the value of some variable)

$P(X = x) \text{ vs } P(X = x|Y = y)$: The probability of X=x can drastically change depending on the knowledge Y=y

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Example: P(lung cancer | smoker) vs

P(lung cancer | smoker, socio-economic status)

Given that the patient is a smoker, does knowing their socio-economic status add further information to the probability of lung cancer?

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Relation between "**joint**", "**conditional**", and "**marginal**" probabilities:

$$P(X, Y) = P(X|Y)P(Y)$$

Bayes' Rule

$A_1, A_2, \ldots, A_n$ are disjoint events forming a **partition** of the sample space and $P(A_i) > 0,\ \forall A_i$. Then, for any event $B,\ P(B) > 0$, Bayes' rule states:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)}$$

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$$
\begin{aligned}
P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} &= \frac{P(A_i)P(B|A_i)}{P(B)} \\
&= \frac{P(A_i)P(B|A_i)}{P(A_1 \cap B) + \cdots + P(A_n \cap B)} \\
&= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)}
\end{aligned}
$$

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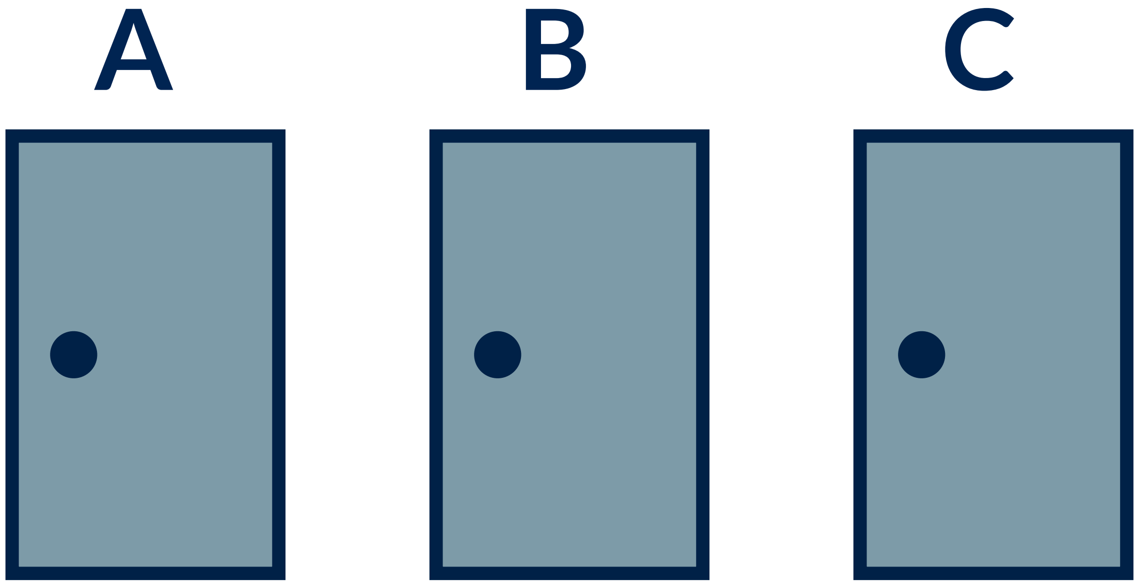
Note: For random variables, we often write $P(X, Y)$, instead of $P(X \cap Y)$

Monte Hall Problem & Application of Bayes' Rule



X = Door chosen by player

Y = Door hiding the car

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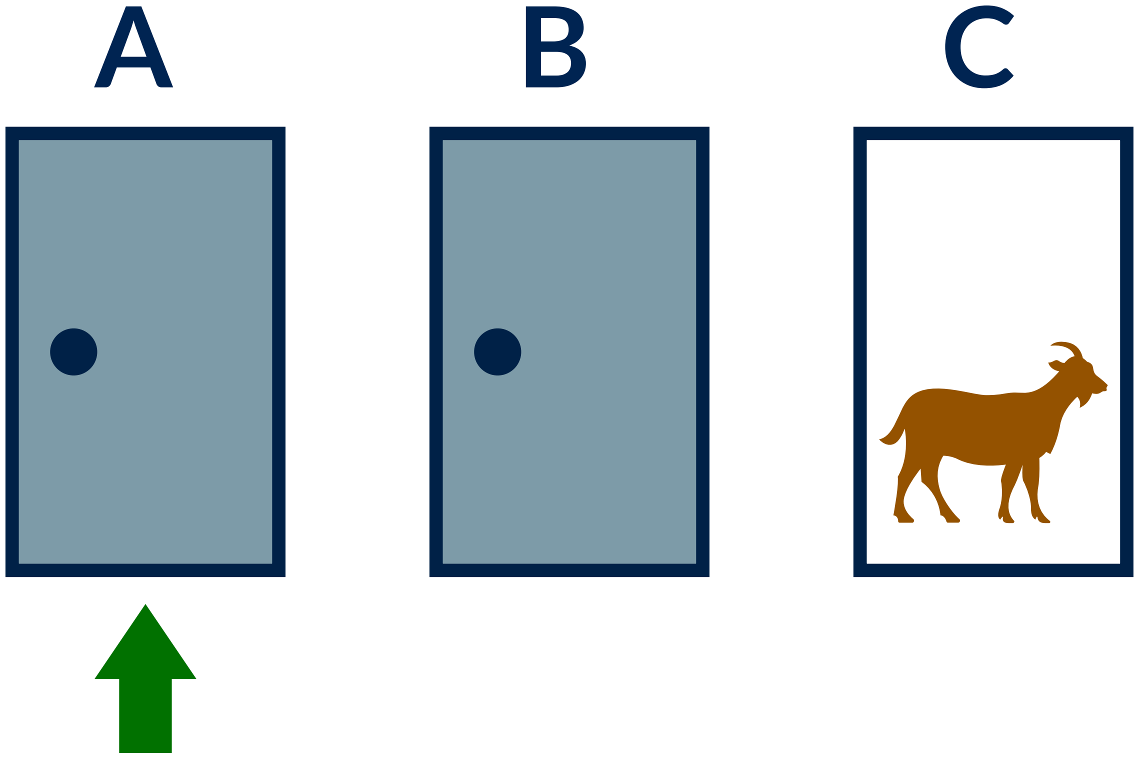

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Prove that switching doors improves our chance of winning the car.

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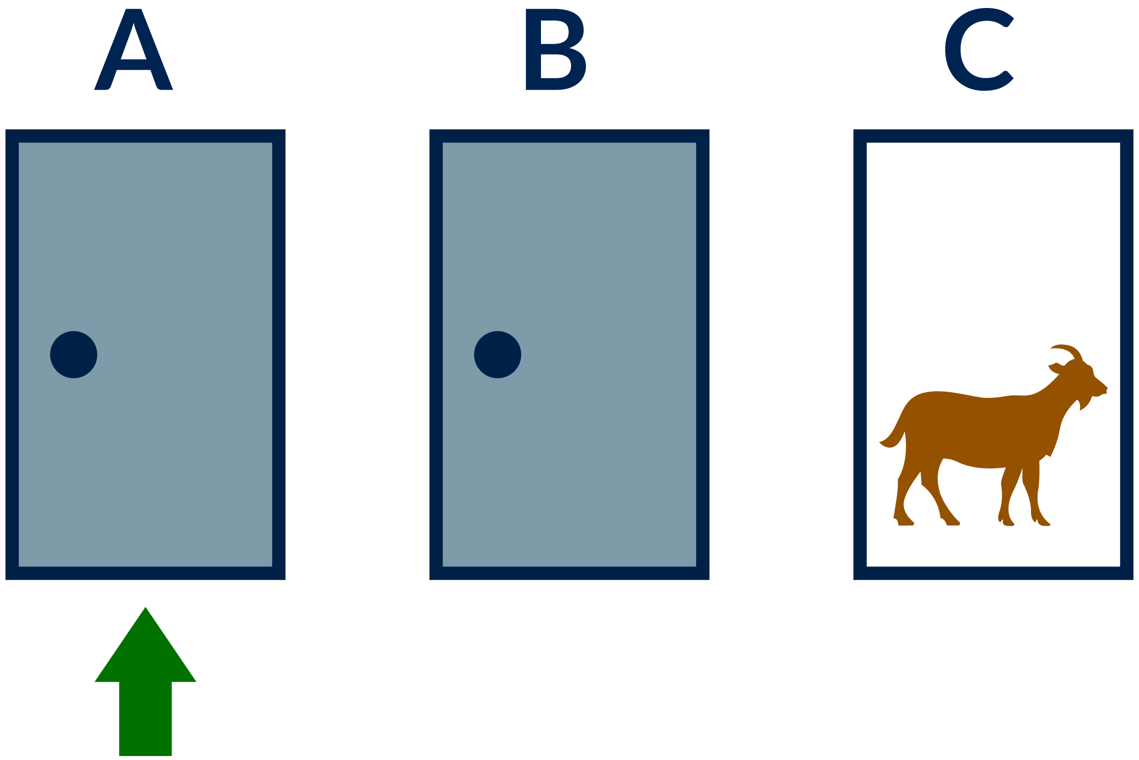

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Prove that switching doors improves our chance of winning the car.

Note the assumptions:

1. The host will not open the door we have chosen
2. **The host will never open a door with a car behind**
3. Given a choice of doors, the host will choose at **random** (whilst 2)
4. Given no info, the car is equally likely to be behind any door

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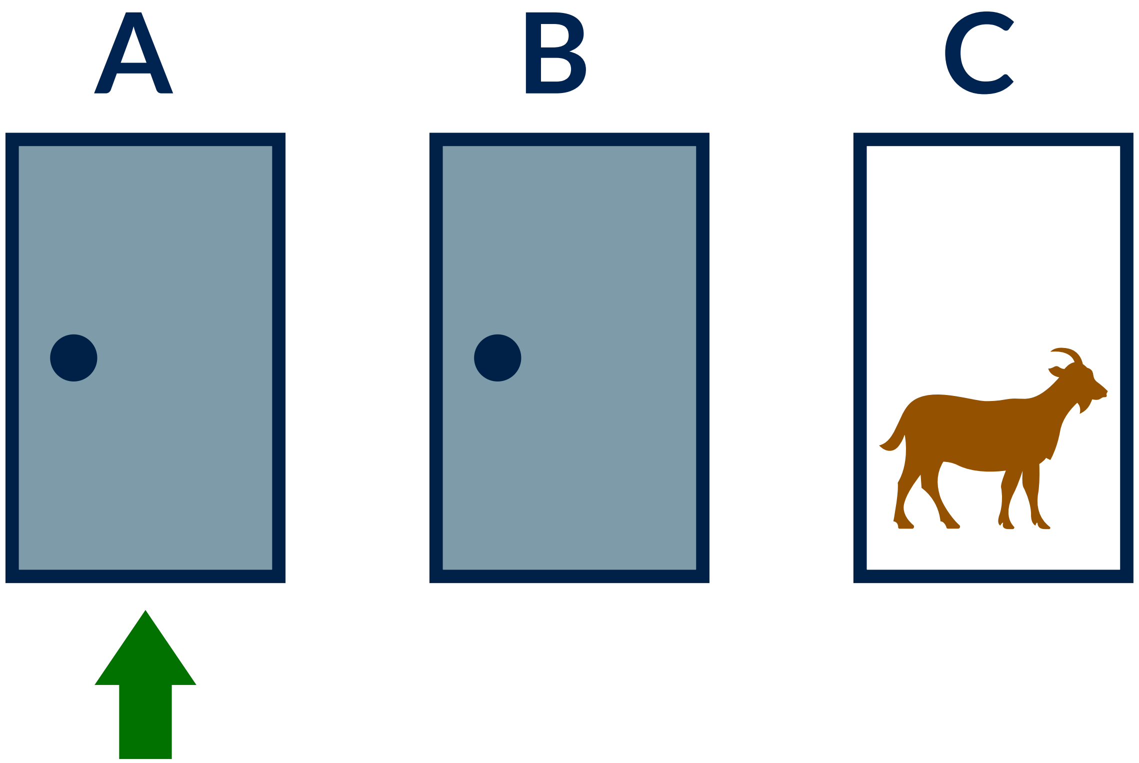

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Prove that switching doors improves our chance of winning the car.

Need to show (given the we have selected A and host has shown us C):

$$P(Y = A | X = A, Z = C) < P(Y = B | X = A, Z = C)$$

Is the car more likely to be behind B than A, i.e. switching improves our chance.

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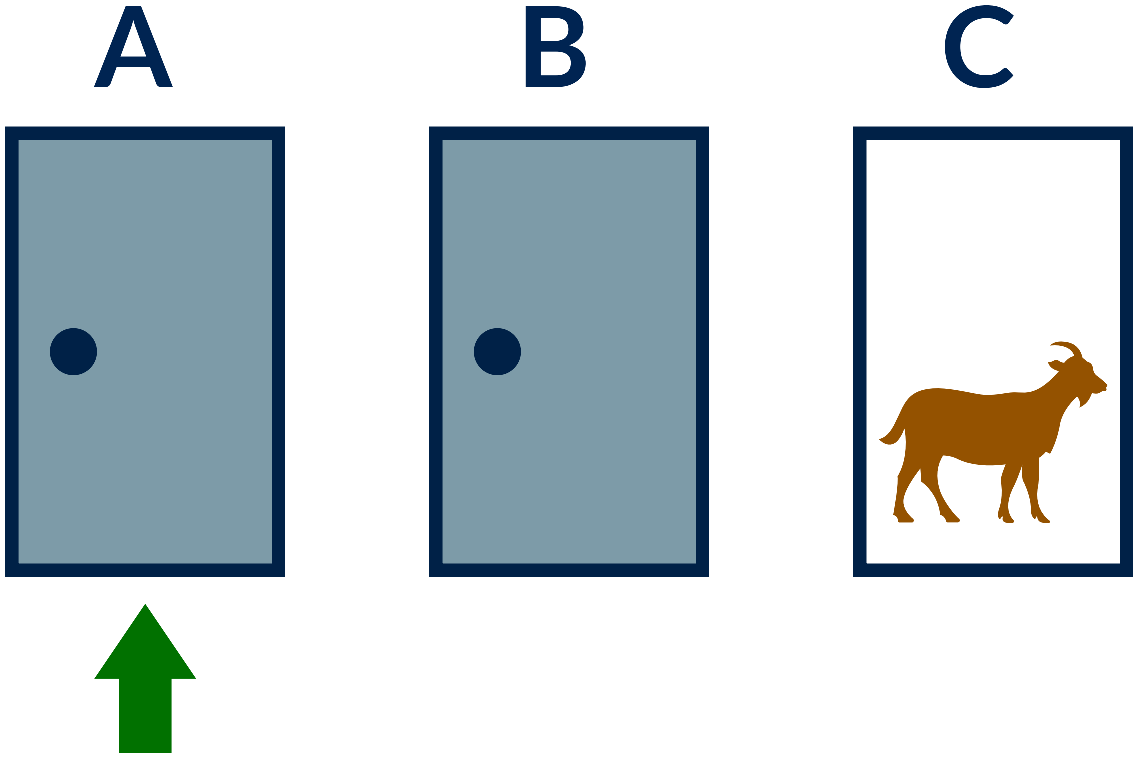

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$$P(Y = A|X = A, Z = C) = \frac{P(Z = C|X = A, Y = A)P(Y = A|X = A)}{P(Z = C|X = A)}$$

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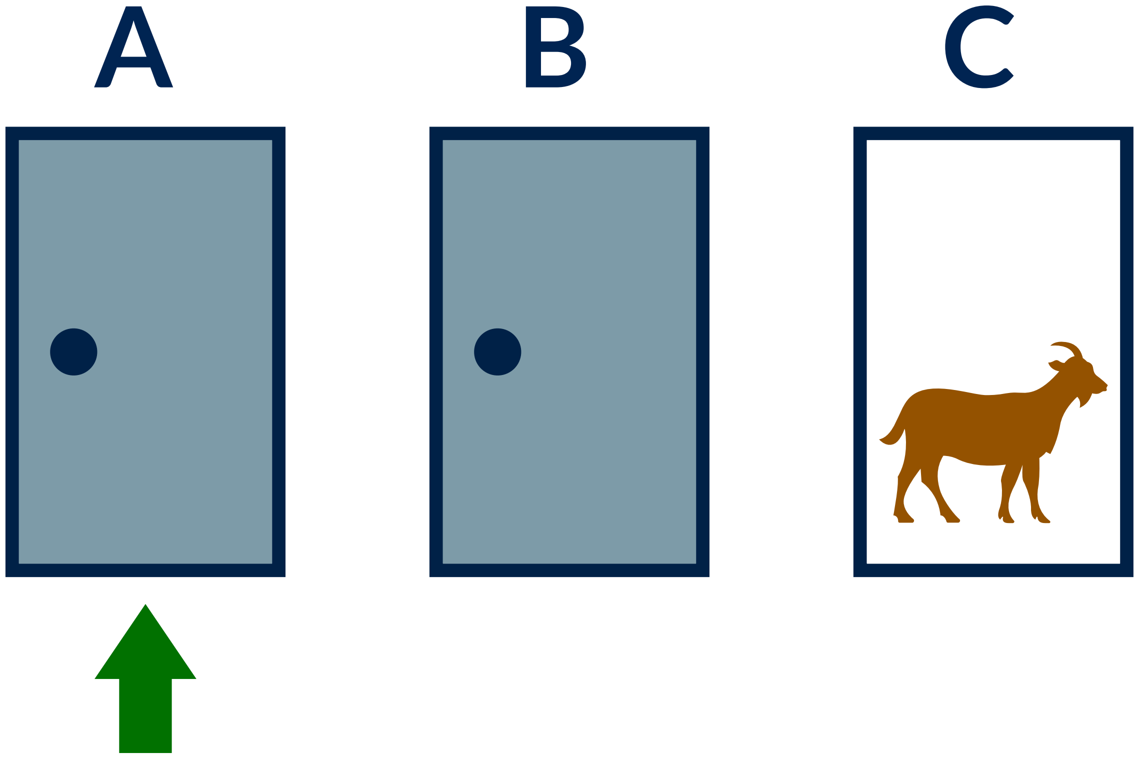

X = Door chosen by player

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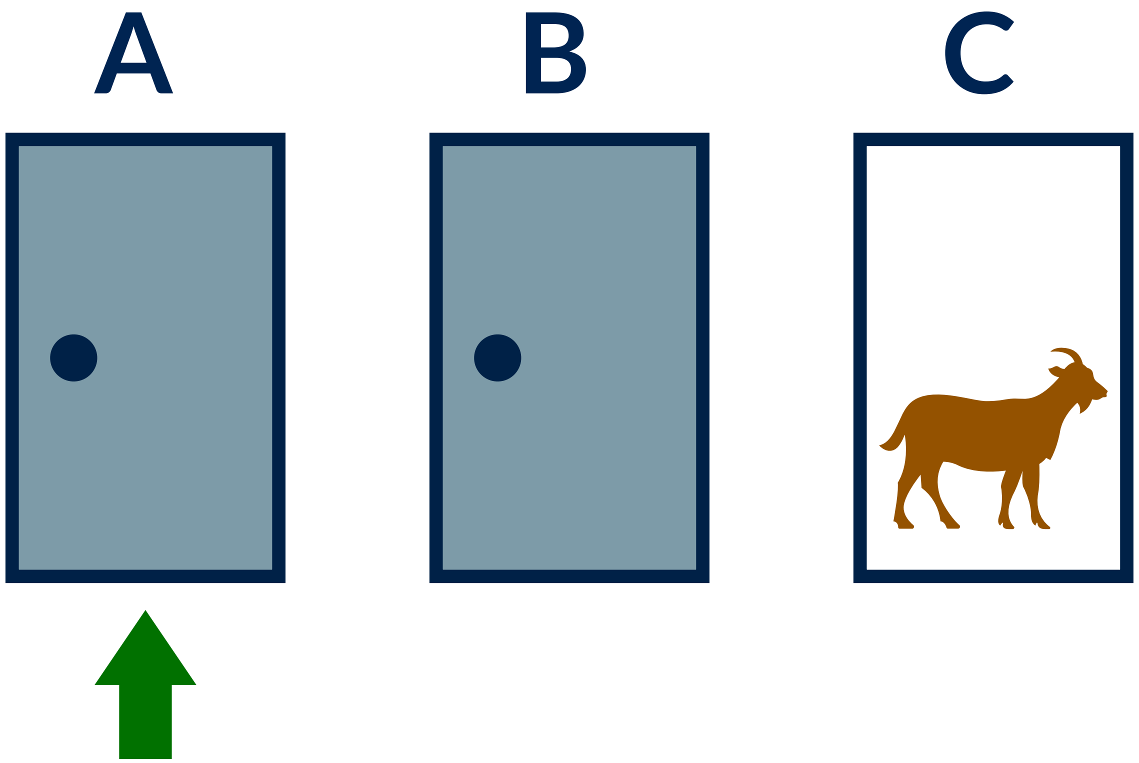
$$P(Y = A|X = A, Z = C) = \frac{\overset{1/2}{\boxed{P(Z = C|X = A, Y = A)}}P(Y = A|X = A)}{P(Z = C|X = A)}$$

Given we choose A (X=A), and the car is in A (Y=A), then the host is allowed to choose either B or C, as neither has the car behind it.

Since the host choses randomly (assumption 3), we get 1/2.

Monte Hall Problem & Application of Bayes' Rule



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$$P(Y = A|X = A, Z = C) = \frac{P(Z = C|X = A, Y = A)\overset{1/3}{\boxed{P(Y = A|X = A)}}}{P(Z = C|X = A)}$$

Given we choose A (X=A), what is the probability that the car is behind A?

With no further information, this is equal to 1/3.

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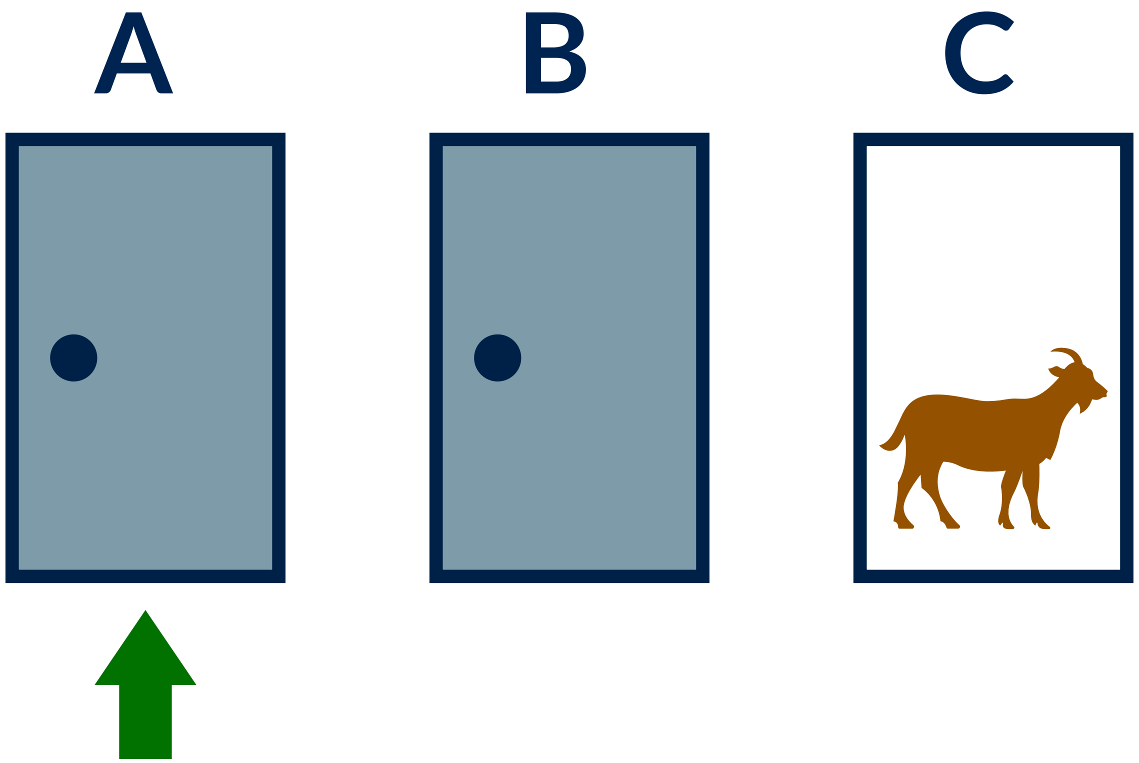

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$$P(Y=A|X=A,Z=C)=\frac{P(Z=C|X=A,Y=A)P(Y=A|X=A)}{\boxed{P(Z=C|X=A)}\ \text{1/2}}$$

$$P(Z=C|X=A)\overset{\text{Total law of prob}}{=}\sum_{d=A,B,C}P(Z=C,Y=d|X=A)\overset{\text{Product rule}}{=}\sum_{d=A,B,C}P(Z=C|X=A,Y=d)P(Y=d)$$

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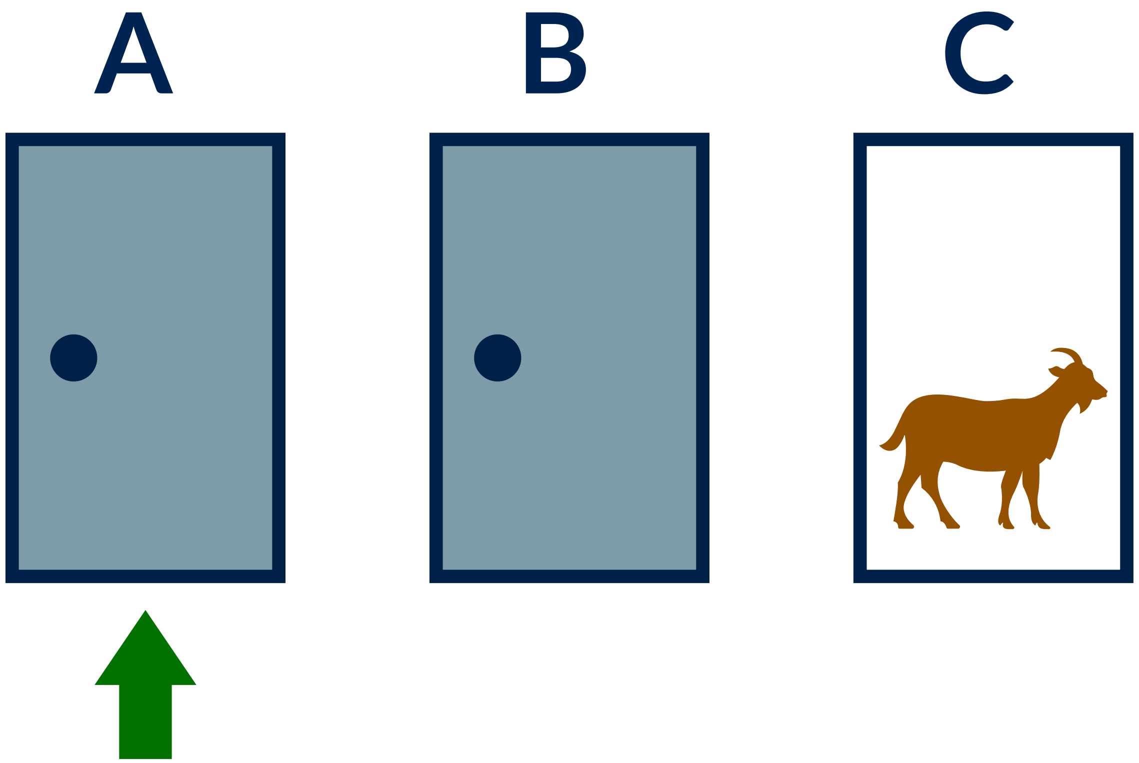

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1/2

Total law of prob

Product rule

$$P(Z=C|X=A) = \sum_{d=A,B,C} P(Z=C,Y=d|X=A) = \sum_{d=A,B,C} P(Z=C|X=A,Y=d)P(Y=d)$$

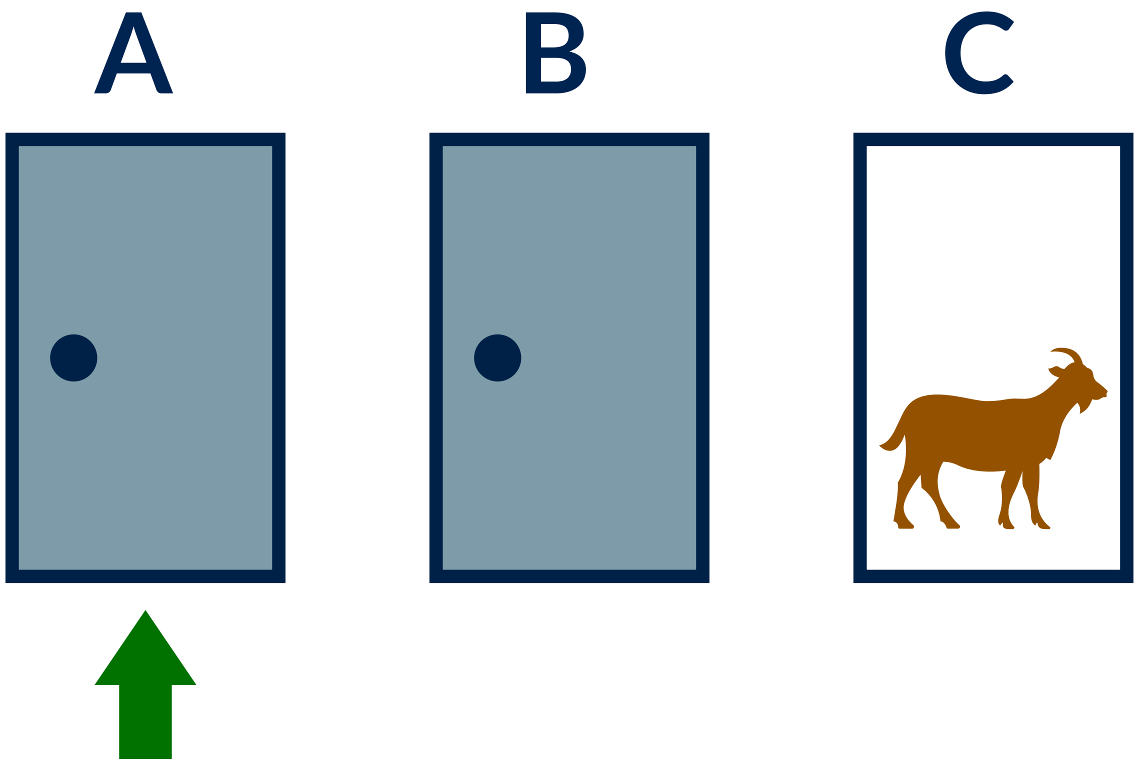
$$= \frac{1}{3}\Big(P(Z=C|X=A,Y=A) + P(Z=C|X=A,Y=B) + P(Z=C|X=A,Y=C)\Big)$$

1/2 as above

1: Given we chose A and car is behind B, host is **forced** to choose C (Assumption 2)

0: Given we chose A and car is behind C, the host cannot choose C (Assumption 2)

Monte Hall Problem & Application of Bayes' Rule



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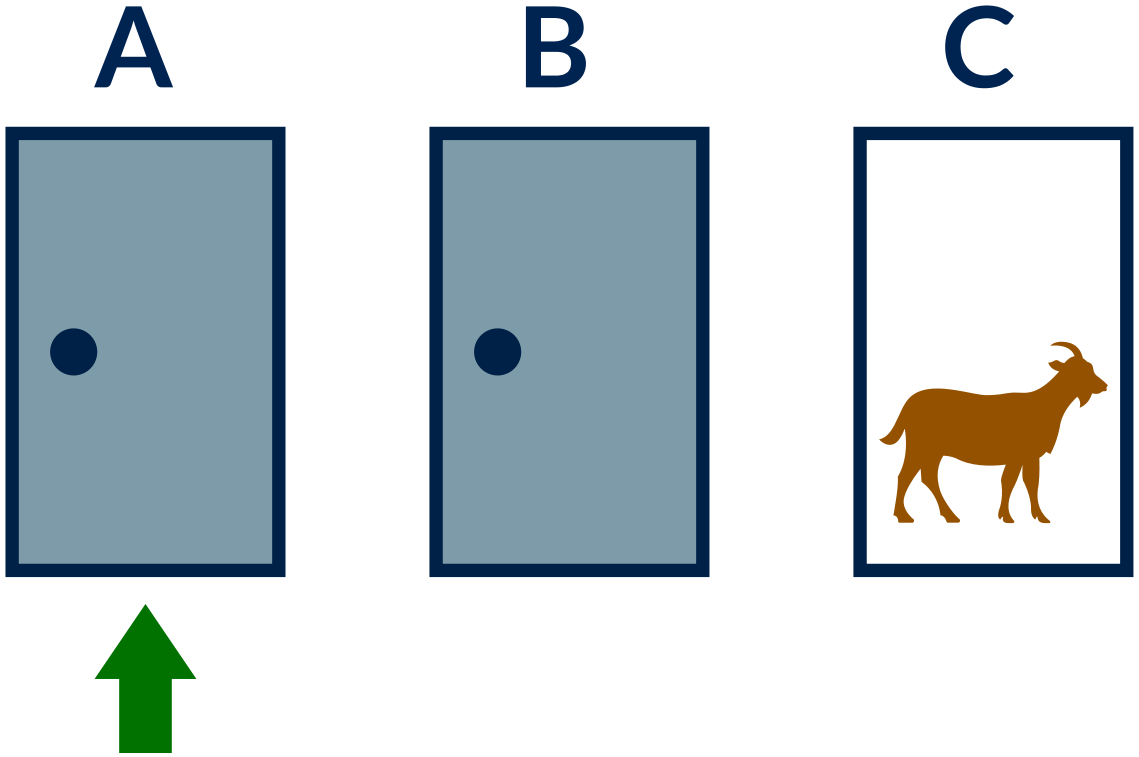
$$P(Y=A|X=A,Z=C)=\frac{\overset{1/2}{P(Z=C|X=A,Y=A)}\overset{1/3}{P(Y=A|X=A)}}{P(Z=C|X=A)\ \ 1/2}$$

Monte Hall Problem & Application of Bayes' Rule



X = Door chosen by player

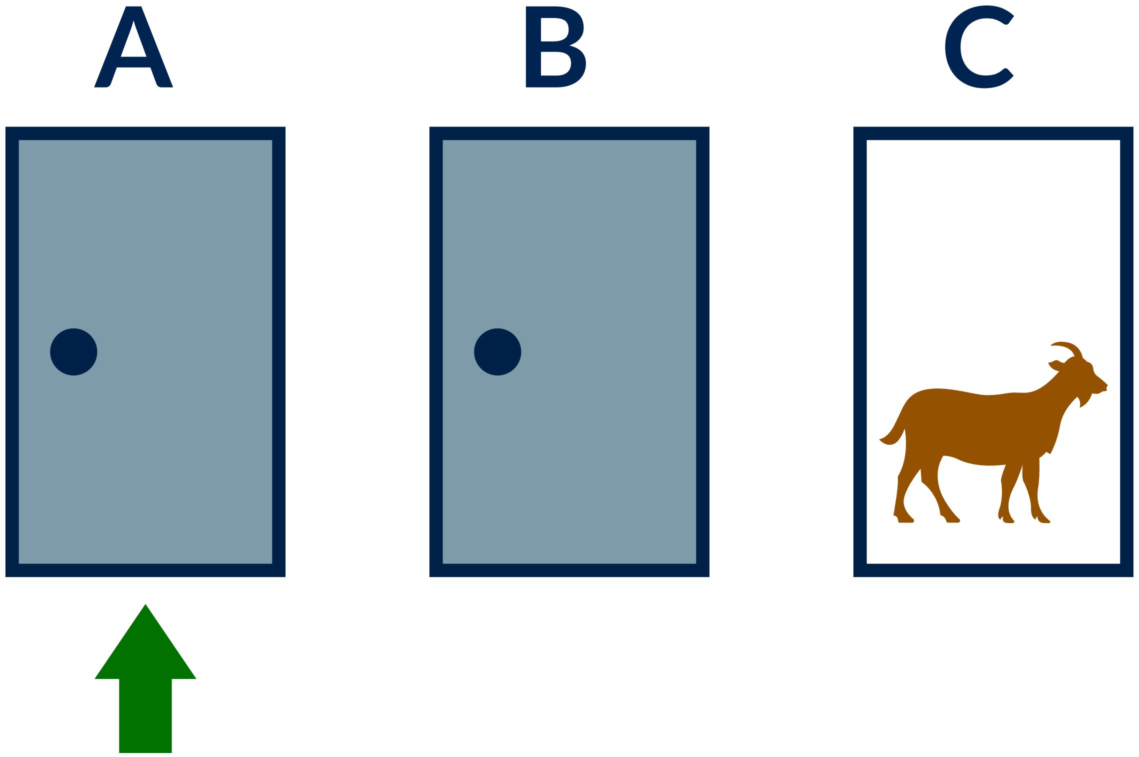
Y = Door hiding the car

Z = Door opened by host

$$P(Y=A|X=A,Z=C)=\frac{\overset{1/2}{P(Z=C|X=A,Y=A)}\,\overset{1/3}{P(Y=A|X=A)}}{P(Z=C|X=A)\quad 1/2}$$

$$\begin{aligned}P(Y=B|X=A,Z=C)&=1-P(Y=A|X=A,Z=C)-P(Y=C|X=A,Z=C)\\&=1-\frac{1}{3}-0=2/3\end{aligned}$$

Monte Hall Problem & Application of Bayes' Rule



X = Door chosen by player

Y = Door hiding the car

Z = Door opened by host

Importance: Incorporating knowledge about the process that generated the data. The first step towards **causal inference**.

'Host could have opened', 'he was forced to open', 'randomly opened', 'about to open', …

Independence

X and Y are independent events: $P(X,Y) = P(X)P(Y)$

Equivalently: $P(X|Y) = P(X)$ (where $P(Y)$ is non-zero, otherwise $P(X|Y)$ not defined)

Conditional independence: $P(X,Y|Z) = P(X|Z)P(Y|Z)$

Equivalently: $P(X|Y,Z) = P(X|Z)$ (again, for $P(Y,Z)$ non-zero)

Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (p1, p2 = 0.5)

Consider 3 events:

H1 = first coin is a head

H2 = second coin is a head

J = the two tosses have the same results

	H	T
H	✗	
T		✗

Independence

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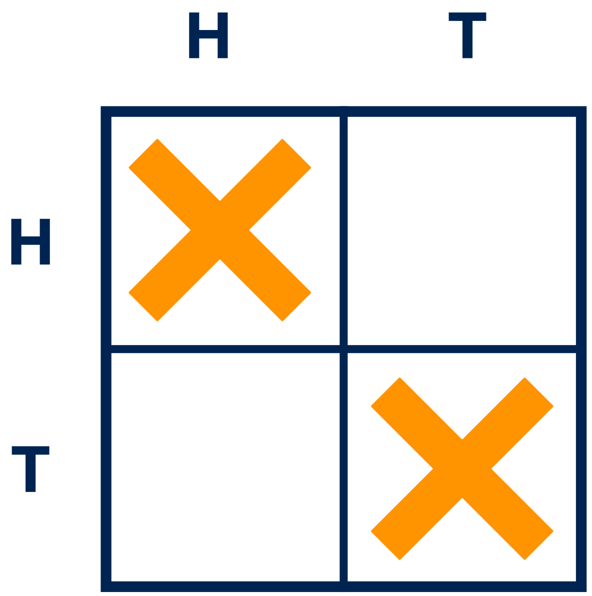
Example: 2 independent fair coin tosses (p1, p2 = 0.5)
H1 & H2: independent coin tosses
$P(H1,H2) = P(H1|H2)P(H2) = 0.5 \times 0.5 = P(H1)P(H2)$



Independence

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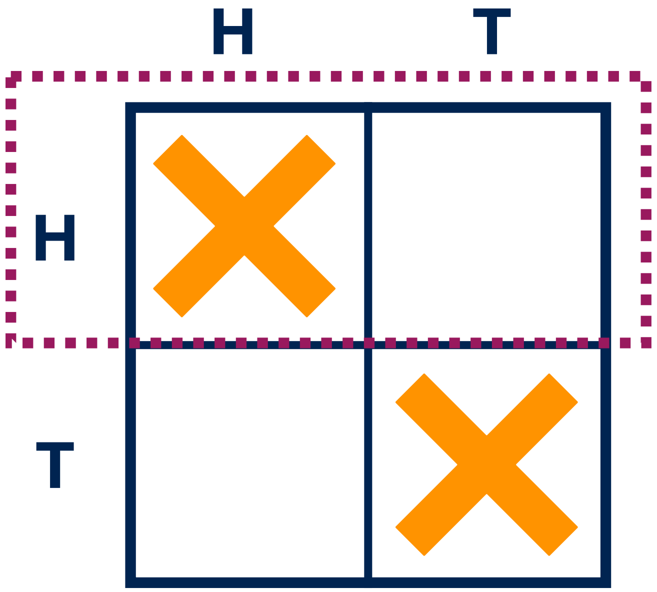
Example: 2 independent fair coin tosses (p1, p2 = 0.5)
H1 & H2: independent coin tosses
P(H1,J) = P(J | H1)P(H1) =
Given H1, what is the probability of J
(i.e second toss also being a head)
So: P(J | H1) = 0.5



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Example: 2 independent fair coin tosses (p1, p2 = 0.5)

H1 & H2: independent coin tosses

$P(H1,J) = P(J \mid H1)P(H1) = 0.5 \times 0.5 = P(J)P(H1)$

Given H1, what is the probability of J

(i.e second toss also being a head)

So: $P(J \mid H1) = 0.5$

	H	T
H	✗	
T		✗

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Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (p1, p2 = 0.5)
H1 & H2: independent coin tosses
P(H2,J) = P(J | H2)P(H2) = 0.5 x 0.5 = P(J)P(H2)
So pair-wise independent. BUT …

	H	T
H	✗	
T		✗

Independence

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Independence of several events:

Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses ($p1, p2 = 0.5$)
H1 & H2: independent coin tosses
$P(H1,H2,J) = P(H1 \mid H2,J)\ P(H2,J) = 1 \times 0.25 = 0.25$

	H	T
H	✗	
T		✗

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Independence of several events:

Remark: Pairwise independence does not imply independence

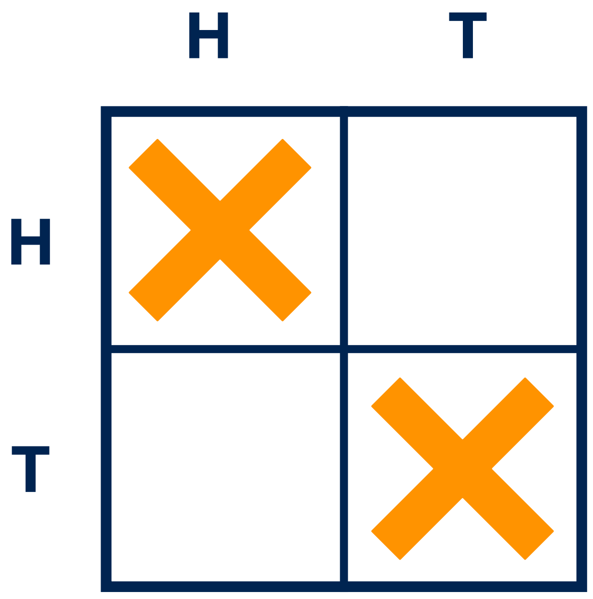
Example: 2 independent fair coin tosses (p1, p2 = 0.5)

H1 & H2: independent coin tosses

P(H1,H2,J) = P(H1 | H2,J) P(H2,J) = 1 x 0.25 = 0.25

However, P(H1)P(H2)P(J)=0.5x0.5x0.5=0.125 $\neq$

i.e. not jointly independent



Expected values

The probability distribution of a random variable X provides us with probabilities of all possible values of X.

Summarise information, with some loss of information, represented by:
The **expected value** or **mean**:

$$\mathbb{E}[X] = \sum_{x} x \ P(X = x)$$

For a dice: (1x1/6) + (2x1/6) + (3x1/6) + (4x1/6) + (5x1/6) + (6x1/6) = 3.5

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The expected value of any function of X, e.g. g(x):

$$\mathbb{E}[g(X)] = \sum_x g(x) \; P(X = x)$$

Dice: (1x1/6) + (4x1/6) + (9x1/6) + (16x1/6) + (25x1/6) + (36x1/6) = 15.17

Expected values

The probability distribution of a random variable X provides us with probabilities of all possible values of X.

Summarise information, with some loss of information, represented by:
The **expected value** or **mean**:

$$\mathbb{E}[X] = \int x \ P(x) dx$$

for a continuous variable X.

Variance

The **variance** of a random variable X, denoted Var(X) or σ_X^2:

$$var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

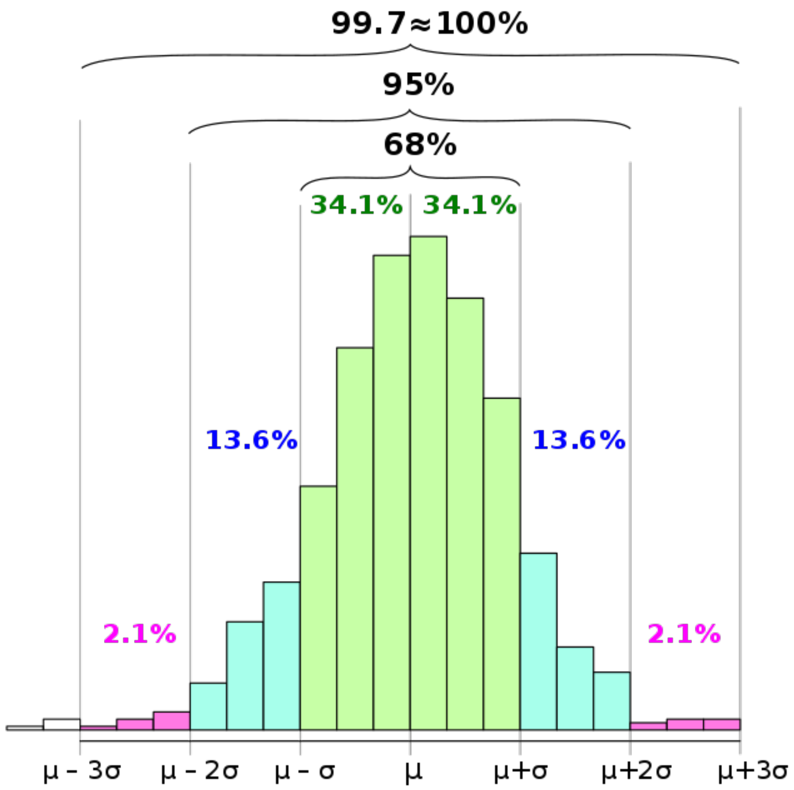
and can be calculated as

$$var(X) = \sum_x (X - \mathbb{E}[X])^2 p_X(x)$$

(Integral of continuous variables), and measure how "spread out" the values of X in a data set are relative to their mean.

The **standard deviation** σ_X (has the same units as X).

For a normal distribution, ~2/3 of the population values of X fall within one σ_X, 95% fall between 2 σ_X, etc.



Covariance

The degree to which two random variables X and Y co-vary (degree associated):

$$\sigma_{XY} = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

and measures a specific way X and Y co-vary, i.e., linearly. When normalised, it yields the correlation coefficient (Pearson correlation):

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

a dimensionless quantity between -1 and 1.

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a dimensionless quantity between -1 and 1.

When X and Y are independent, then $\rho_{XY} = 0$.
The reverse is not true!
(e.g. ρ_{XY} may be zero, but not linear-correlation, hence dependence exists.
This requires more complex methods of demonstrating if $P(Y|X) = P(Y)$)

Anscombe's Quartet

Group of 4 datasets with nearly identical simple descriptive statistical properties:

- Mean and sample variance of X
- Mean and sample variance of Y
- Correlation between X and Y
- Linear regression line (coefficient the same up to 2 or 3 decimal places)
- R^2 coefficient

A note on R^2: A measure for goodness-of-fit

$$R^2 = 1 - \frac{\sum_i (y_i - f_i)^2}{\sum_i (y_i - \bar{y})^2} \;, \; y_i = f(x_i) \;, \; \bar{y} = \frac{1}{n} \sum_i y_i$$

If the fit y=f(x) is a perfect fit, the numerator is zero, $R^2 = 1$, and
$R^2 = 0$ implies the fit f(x) is no better than baseline average $\bar{y}$.
Negative values corresponds to models worse than the baseline average.

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- Mean and sample variance of X
- Mean and sample variance of Y
- Correlation between X and Y
- Linear regression line (coefficient the same up to 2 or 3 decimal places)
- R^2coefficient

Yet, very different distributions, which can be observed by plotting the graphs

Same Pearson correlation, but,
different dependence structure
(X causes Y, but in different ways)

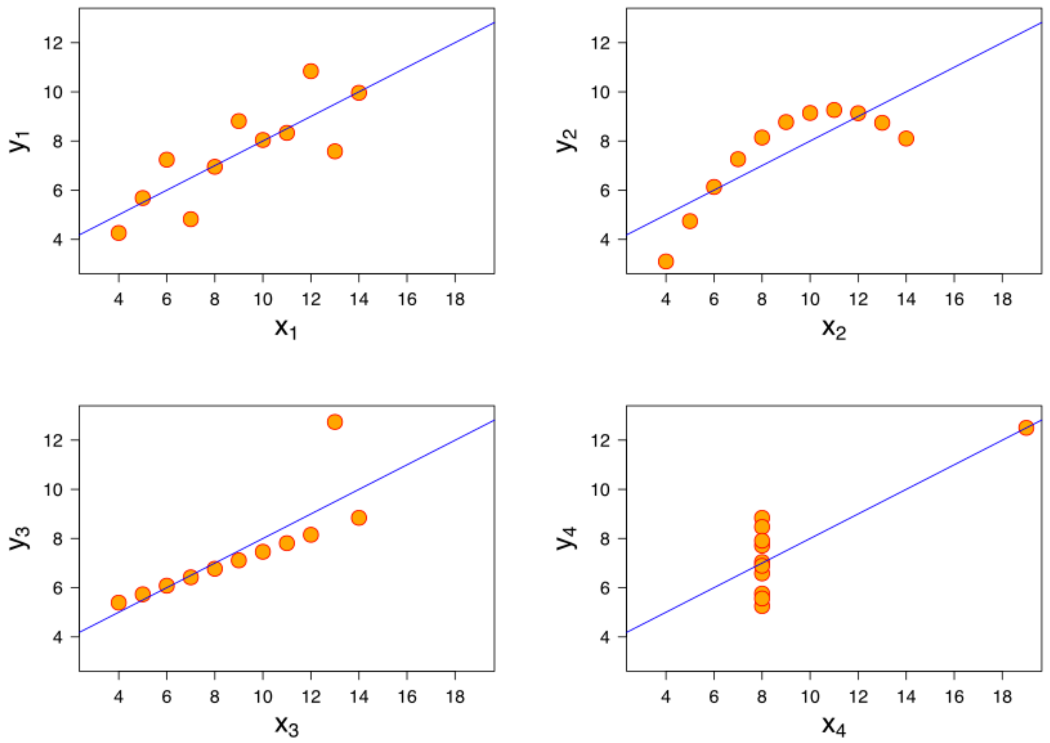

Figure from WikiPedia

THE UNIVERSITY
of EDINBURGH

Methods for Causal Inference
Lecture 2: Basics of probability

Ava Khamseh

School of Informatics
2025-2026