THE UNIVERSITY of EDINBURGH

Methods for Causal Inference
Lecture 6: Instrumental variable method

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School of Informatics

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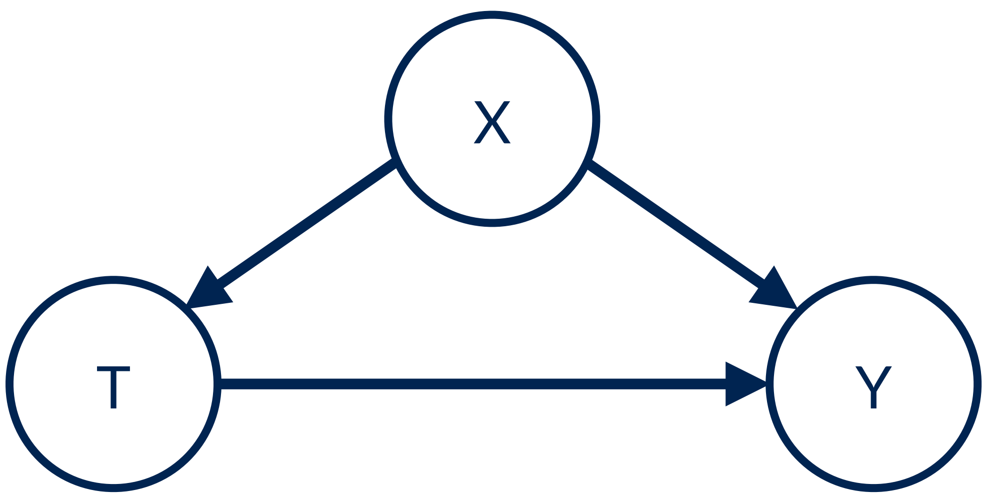
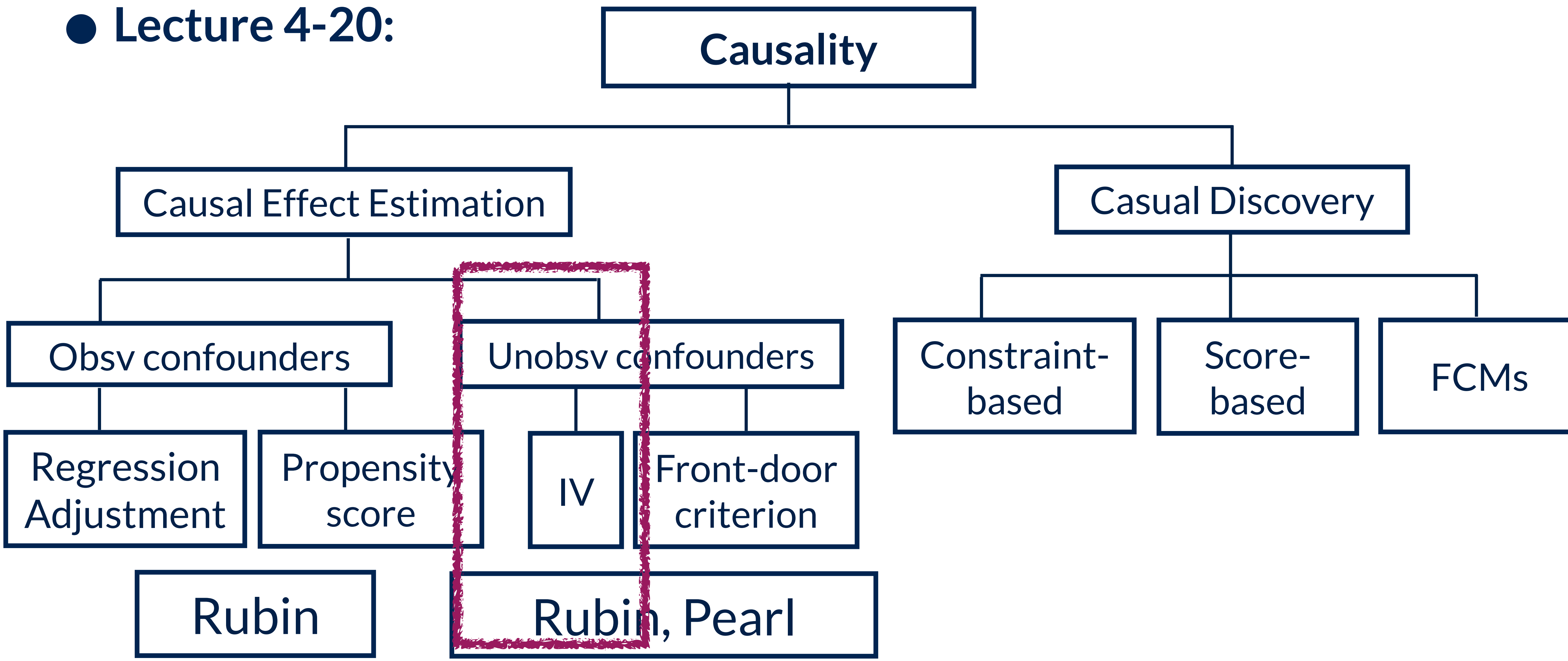
So far …

Causal inference with observed confounders



Overview of the course

- **Lecture 1**: Introduction & Motivation, why do we care about causality? Why deriving causality from observational data is non-trivial.
- **Lecture 2:** Recap of probability theory, variables, events, conditional probabilities, independence, law of total probability, Bayes' rule
- **Lecture 3**: Recap of regression, multiple regression, graphs, SCM
- **Lecture 4-20:**



Randomised Controlled Trials (RCTs)

Randomised Control Trials (RCT): Subjects are assigned at random to various groups (treatment or control)

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- Randomisation may influence participation and behaviour

Randomising an instrument

Causal inference from studies in which subject have a final choice

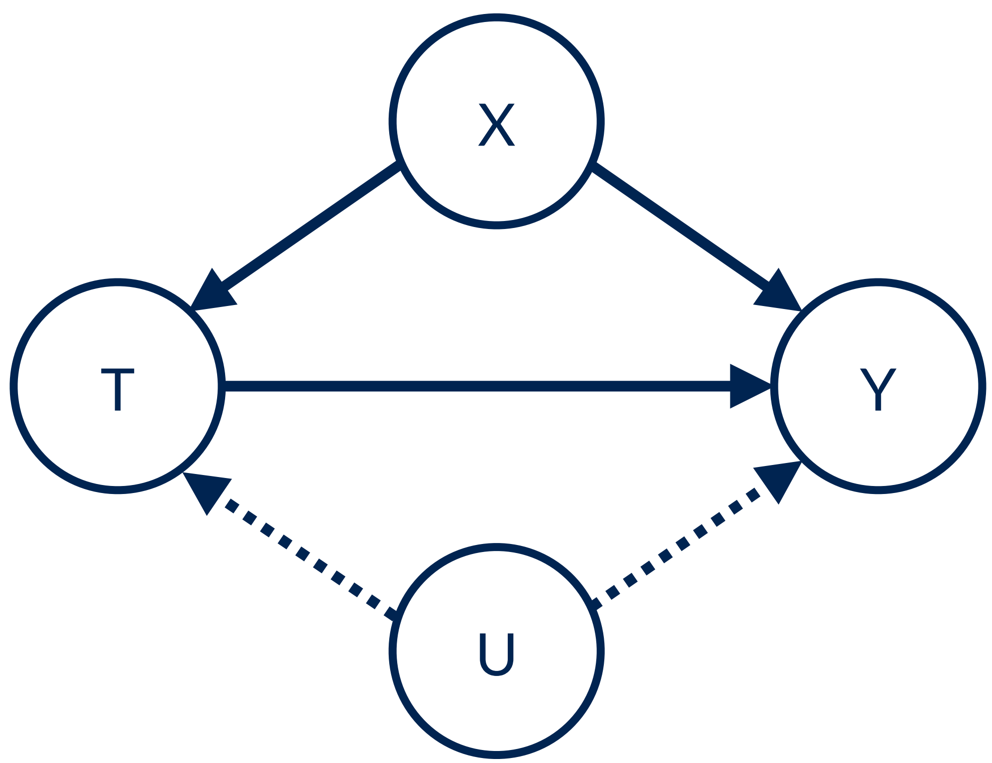
Randomisation is confined to an indirect **instrument** that encourages or discourage participation in treatment or control programmes.

(However, imperfect compliance poses a problem, e.g., subjects that declined taking the drug are precisely those who would have responded adversely. So experiment might conclude the drug is more effective than it actually is.
-> more complex methods, e.g. bounds)

Instrumental Variable

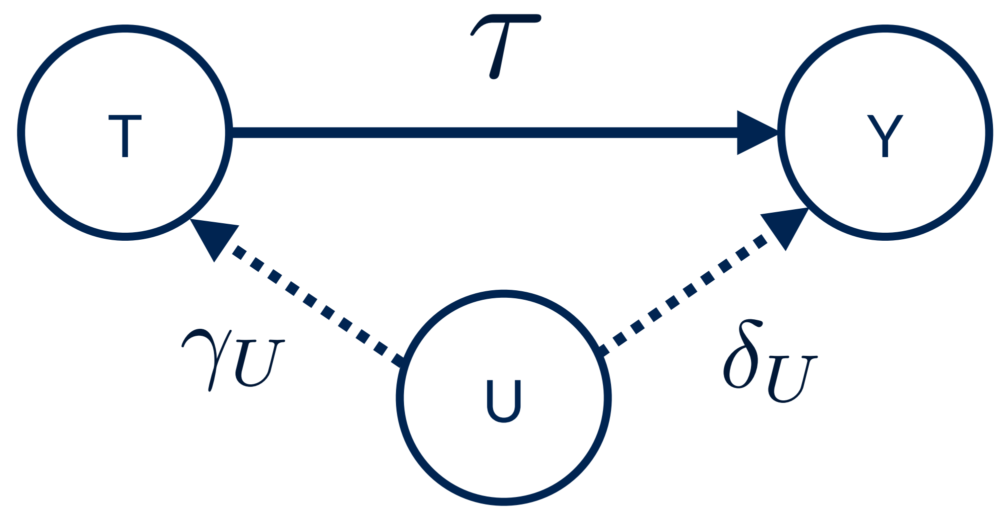
Unobserved confounders (U), **violates unconfoundedness,** i.e. conditioning on X alone, would not results in a randomised treatment assignment

Unconfoundedness is fundamentally unverifiable



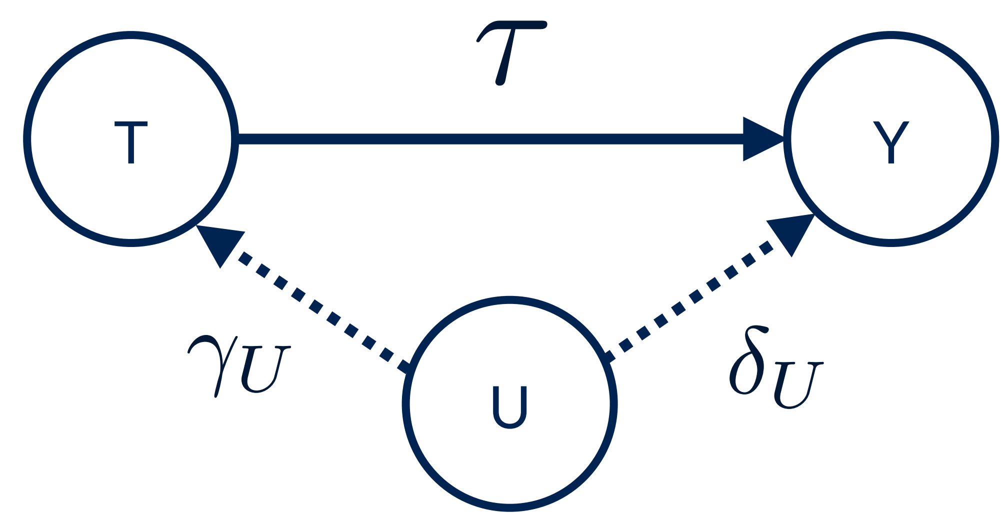
Rubin 1996

Naive regression leads to bias



$$Y = \tau T + \delta_U U$$
$$T = \gamma_U U$$

Naive regression leads to bias



What happens if we naively perform a linear regression of Y on T:

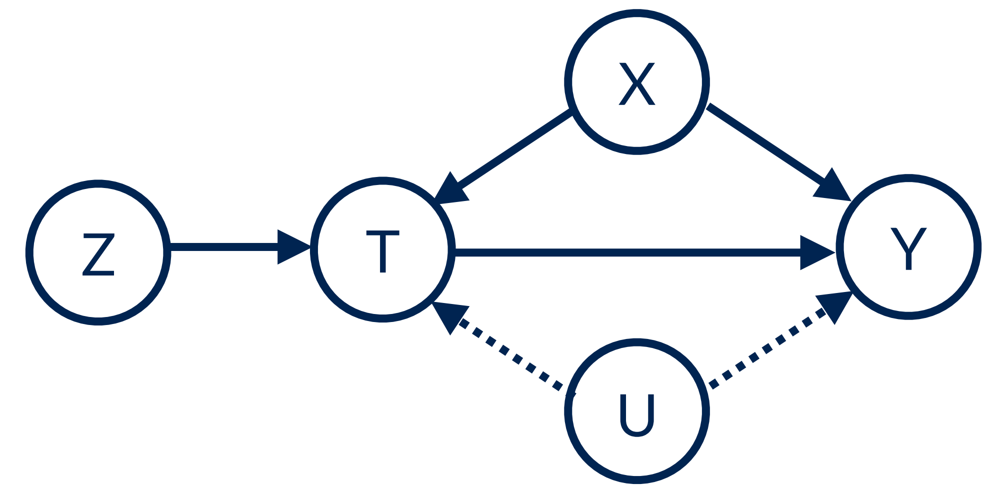
$$Y = \tau T + \delta_U U$$
$$T = \gamma_U U$$

$$\frac{\mathrm{Cov}[T, Y]}{\mathrm{Var}[T]} = \frac{\tau \mathrm{Var}[T] + \gamma_U \delta_U \mathrm{Var}[U]}{\mathrm{Var[T]}} = \tau + \frac{\gamma_U \delta_U \mathrm{Var}[U]}{\mathrm{Var[T]}} = \tau + \frac{\delta_U}{\gamma_U}$$

causal term

Bias term

Instrumental Variable example



- **Example 1:**
  - T: smoking during pregnancy
  - Y: birthweight
  - X: parity, mother's age, weight, …
  - U: Other unmeasured confounders

- Randomise Z (intention-to-treat): either receive encouragement to stop smoking (Z=1), or receive usual care (Z=0)
- Intention-to-treat analysis gives causal effect estimator of encouragement z on outcome y:

$$\mathbb{E}(y|z=1) - \mathbb{E}(y|z=0)$$

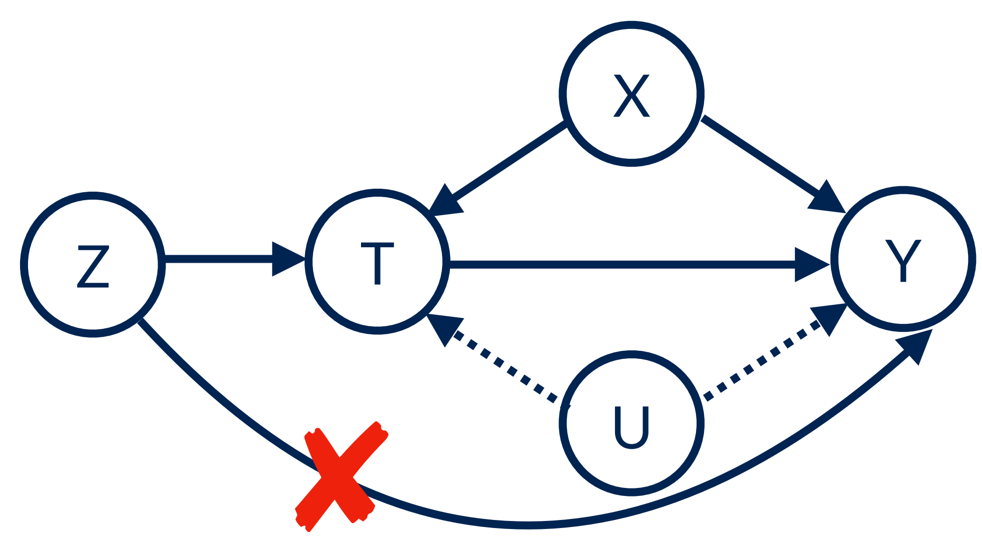
- What can we say about the causal effect of smoking itself?

Instrumental Variable assumptions

- **SUTVA**: Potential outcomes for each individual i are unrelated to the treatment status of other individuals:

$$Y^{(i)}(\mathbf{Z}, \mathbf{T}) = Y^{(i)}(Z^{(i)}, T^{(i)}) \ , \ |\mathbf{Z}| = |\mathbf{T}| = N \text{ individuals}$$

- Non-zero average/relevant: Treatment assignment Z associated with the treatment $\mathbb{E}\left[\left(T^{(i)}|z=1\right) - \left(T^{(i)}|z=0\right)\right]$
- Treatment assignment Z is random (Z and Y do not share a cause).



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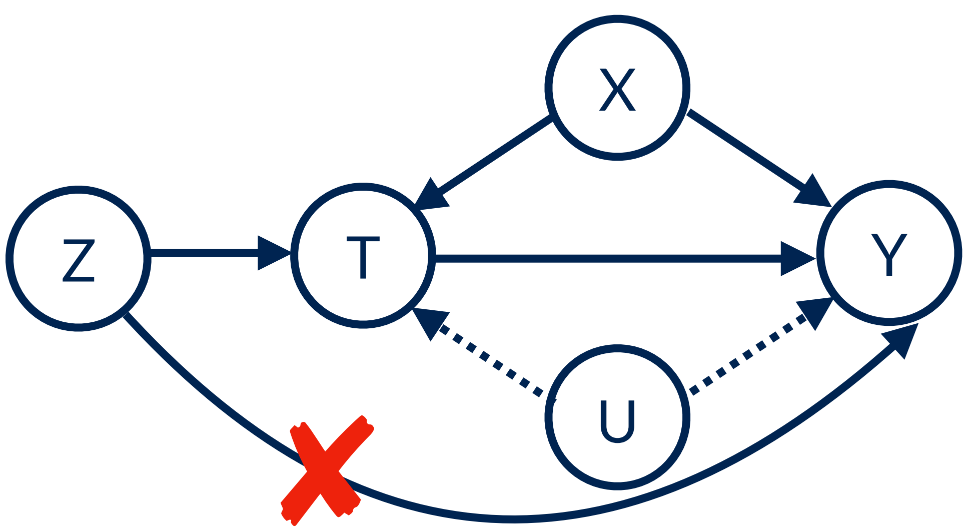
- Treatment assignment Z is random (Z and Y do not share a cause).

$$\left(Y^{(i)}|z=1,t\right) = \left(Y^{(i)}|z=0,t\right)$$

- **Exclusion Restriction**: Any effect of Z on Y is via an effect of Z on T, i.e., Z should not affect Y when T is held constant

- **Monotonicity** (increasing encouragement "dose" increases probability of treatment, no defiers):

$$\left(T^{(i)}|z=1\right) \geq \left(T^{(i)}|z=0\right)$$



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Instrumental Variable: Potential values of T

Population	T|z=0	T|z=1	Description
Never-takers	0	0	Causal effect of Z on T is zero, since $\left(T^{(i)} \vert z=1\right) - \left(T^{(i)} \vert z=0\right) = 0$
Compliers	0	1	$\left(T^{(i)} \vert z=1\right) - \left(T^{(i)} \vert z=0\right) = 1$ **causal effect inference**: $\left(Y^{(i)} \vert T^{(i)}=1\right) - \left(Y^{(i)} \vert T^{(i)}=0\right)$
Defiers	1	0	Rule out by **monotonicity**, since $\left(T^{(i)} \vert z=1\right) - \left(T^{(i)} \vert z=0\right) = -1$
Always-takers	1	1	Causal effect of Z on Y is zero, since $\left(T^{(i)} \vert z=1\right) - \left(T^{(i)} \vert z=0\right) = 0$

Notation: T=1 is **not** smoking

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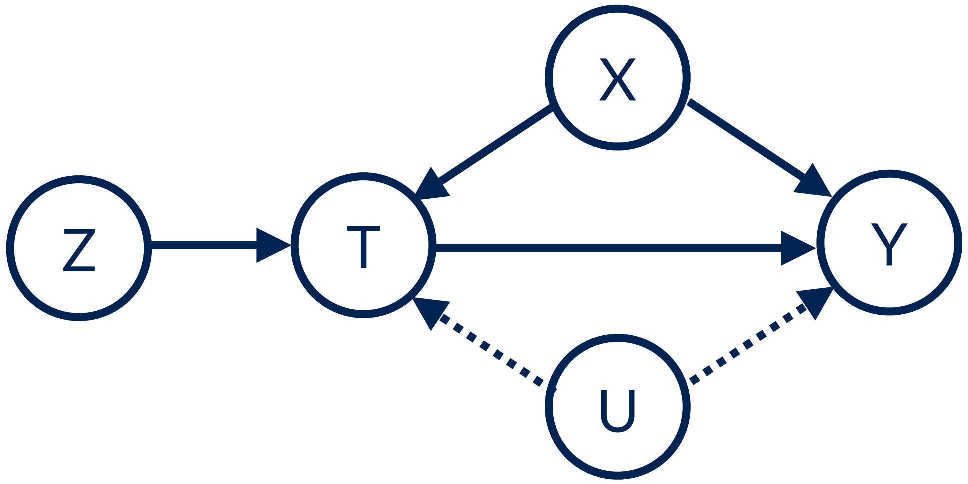
Instrumental Variable: The estimand

Want ATE: $$\mathbb{E}\left[Y_{T=1} - Y_{T=0}\right]$$



"Almost"

Will estimate: $$\boxed{\tau = \frac{\mathbb{E}\left[(Y|z=1) - (Y|z=0)\right]}{\mathbb{E}\left[(T|z=1) - (T|z=0)\right]}}$$

Rubin 1996

Instrumental Variable: The estimand

Want ATE: $$\mathbb{E}\left[\left(Y^{(i)}|t^{(i)}=1\right)-\left(Y^{(i)}|t^{(i)}=0\right)\right]$$

$$\boxed{\tau=\frac{\mathbb{E}\left[(Y|z=1)-(Y|z=0)\right]}{\mathbb{E}\left[(T|z=1)-(T|z=0)\right]}}$$

Derivation:

$$\left(Y^{(i)}|T^{(i)}(z=1)\right)-\left(Y^{(i)}|T^{(i)}(z=0)\right)$$

t is either t=0 or t=1, and exclusion restriction

$$=\left[Y^{(i)}\left(t^{(i)}=1\right)\cdot\left(t^{(i)}|z=1\right)+Y^{(i)}\left(t^{(i)}=0\right)\cdot\left(1-\left(t^{(i)}|z=1\right)\right)\right]$$

$$-\left[Y^{(i)}\left(t^{(i)}=1\right)\cdot\left(t^{(i)}|z=0\right)+Y^{(i)}\left(t^{(i)}=0\right)\cdot\left(1-\left(t^{(i)}|z=0\right)\right)\right]$$

$$=\left(Y^{(i)}\left(t^{(i)}=1\right)-Y^{(i)}\left(t^{(i)}=0\right)\right)\cdot\left(\left(t^{(i)}|z=1\right)-\left(t^{(i)}|z=0\right)\right)$$

Hence, the causal effect of Z on Y for individual i, is the product of the causal effect of Z on T, and, the casual effect of T on Y.

Rubin 1996

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Rubin 1996

Instrumental Variable: The estimand

To continue the derivation, we use the fact that:

$$\mathbb{E}\left[XY\right] = \int\int xy\ p(x,y)dxdy = \int dy\ y\ p(y) \int dx\ x\ p(x|y) = \int dy\ y\ p(y)\mathbb{E}[x|y]$$

and write,

$$\mathbb{E}\left[\left(Y^{(i)}|T^{(i)}(z=1)\right) - \left(Y^{(i)}|T^{(i)}(z=0)\right)\right]$$

0, 1, -1

$$= \mathbb{E}\left[\left(Y^{(i)}\left(t^{(i)}=1\right) - Y^{(i)}\left(t^{(i)}=0\right)\right) \cdot \left(\left(t^{(i)}|z=1\right) - \left(t^{(i)}|z=0\right)\right)\right]$$

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$$\begin{aligned}
&\mathbb{E}\left[\left(Y^{(i)}|T^{(i)}(z=1)\right) - \left(Y^{(i)}|T^{(i)}(z=0)\right)\right] \\
&= \mathbb{E}\left[\left(Y^{(i)}\left(t^{(i)}=1\right) - Y^{(i)}\left(t^{(i)}=0\right)\right)\cdot\left(\left(t^{(i)}|z=1\right) - \left(t^{(i)}|z=0\right)\right)\right] \\
&= \mathbb{E}\left[\left(Y^{(i)}\left(t^{(i)}=1\right) - Y^{(i)}\left(t^{(i)}=0\right)\right) | \left(\left(t^{(i)}|z=1\right) - \left(t^{(i)}|z=0\right)\right) = 1\right]\cdot \\
&\qquad P\left(\left(t^{(i)}|z=1\right) - \left(t^{(i)}|z=0\right) = 1\right) \\
&-\mathbb{E}\left[\left(Y^{(i)}\left(t^{(i)}=1\right) - Y^{(i)}\left(t^{(i)}=0\right)\right) | \left(\left(t^{(i)}|z=1\right) - \left(t^{(i)}|z=0\right)\right) = -1\right]\cdot \\
&\qquad P\left(\left(t^{(i)}|z=1\right) - \left(t^{(i)}|z=0\right) = -1\right)
\end{aligned}$$

0, by restricting to compliers

Rubin 1996

Instrumental Variable: The estimand

$$\frac{\mathbb{E}\left[\left(Y^{(i)}|T^{(i)}(z=1)\right)-\left(Y^{(i)}|T^{(i)}(z=0)\right)\right]}{\mathbb{E}\left[\left(t^{(i)}|z=1\right)-\left(t^{(i)}|z=0\right)\right]}$$

$$=\mathbb{E}\left[\left(Y^{(i)}\left(t^{(i)}=1\right)-Y^{(i)}\left(t^{(i)}=0\right)\right)\middle|\boxed{\left(\left(t^{(i)}|z=1\right)-\left(t^{(i)}|z=0\right)\right)=1}\right]$$

i.e. restricting to *compliers*, the average causal effect of Z on Y is proportional to the average causal effect of T on Y.

$$\boxed{\tau=\frac{\mathbb{E}\left[(Y|z=1)-(Y|z=0)\right]}{\mathbb{E}\left[(T|z=1)-(T|z=0)\right]}}$$

- In this example, Z was randomly assigned as part of the study
- IV can also be randomised in nature (nature randomiser):
  - Mendelian randomisation

Rubin 1996

Instrumental Variable: Mendelian Randomisation

Population genetics:

Z = a DNA variant associated with a particular exposure T

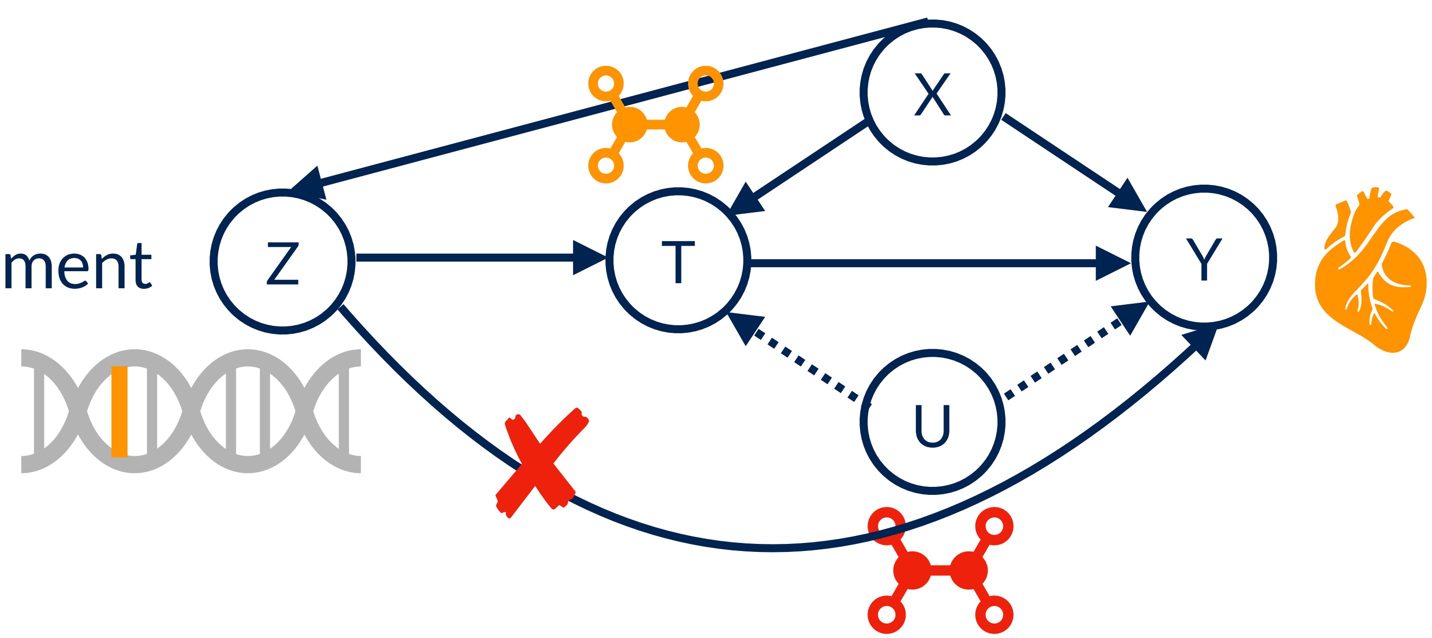
T = exposure, e.g. lipid levels in the blood

Y = heart disease

X = population stratification (might affect Z, need to adjust)

U = unobserved variables affecting both lipid levels and disease

Conditional instrument



Instrumental Variable: Economics

How does price of a product casually affect demand?

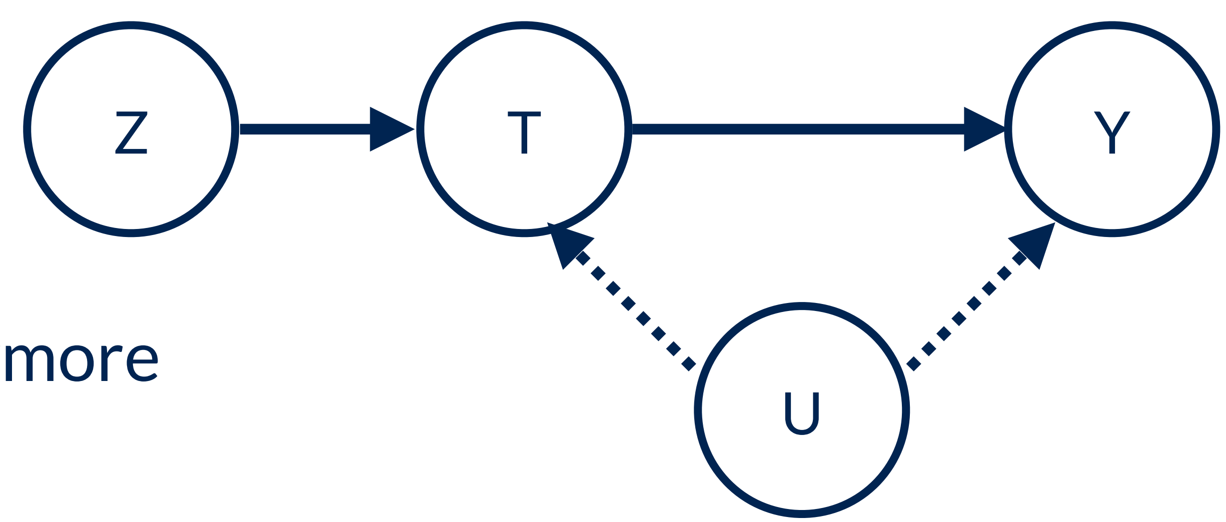
Z = Market supply

T = Price

Y = Demand

U = Factors confounding influencing price and demand
(e.g. tax imposed)

Exclusion restriction requires that market supply does not affect demand
(e.g. COVID-19 toilet paper fiasco!)
(e.g. Pokemon cards)
Also, individuals may not be independent anymore



The Wald Estimator (for binary variables)

$$\boxed{\tau = \frac{\mathbb{E}\left[(Y|z=1) - (Y|z=0)\right]}{\mathbb{E}\left[(T|z=1) - (T|z=0)\right]}}$$

$$\boxed{\hat{\tau} = \frac{\frac{1}{n_{z=1}} \sum_{i \in z=1} Y^{(i)} - \frac{1}{n_{z=0}} \sum_{i \in z=0} Y^{(i)}}{\frac{1}{n_{z=1}} \sum_{i \in z=1} T^{(i)} - \frac{1}{n_{z=0}} \sum_{i \in z=0} T^{(i)}}}$$

IV Estimator: continuous variables case

Linear case:

$$\boxed{\tau = \frac{\mathrm{Cov}(Y, Z)}{\mathrm{Cov}(T, Z)}}$$

$$\boxed{\hat{\tau} = \frac{\widehat{\mathrm{Cov}}(Y, Z)}{\widehat{\mathrm{Cov}}(T, Z)}}$$

Two-Stage Least-squares Estimator

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$$\mathrm{Cov}(Y, Z) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z]$$

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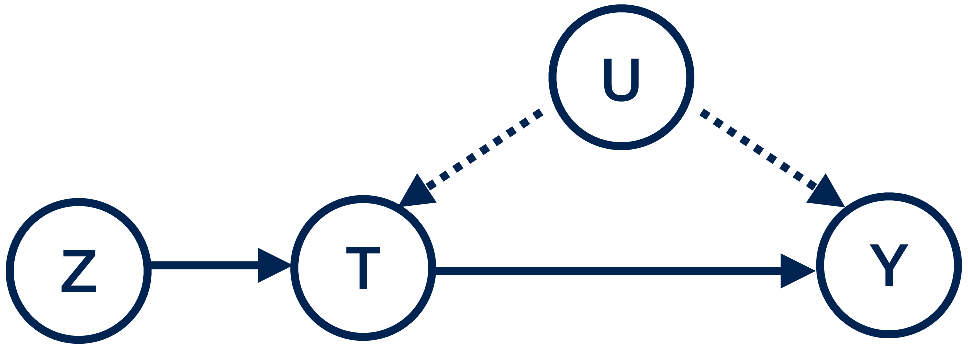


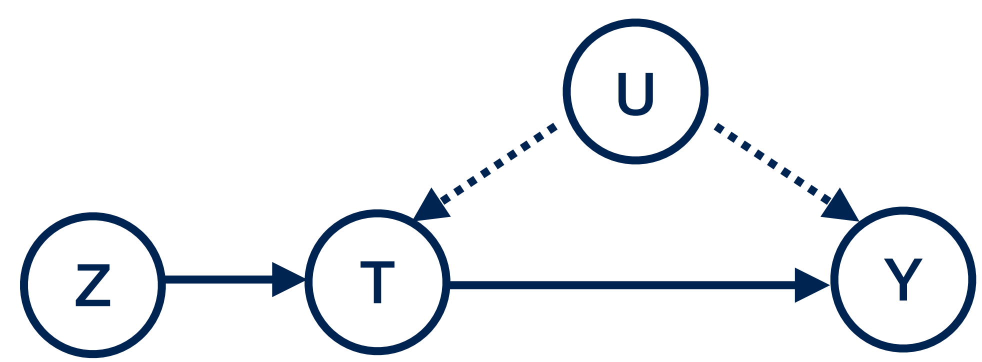
$$Y = \tau T + \delta_U U$$

IV Estimator: continuous variables case

$$
\begin{aligned}
\mathrm{Cov}(Y, Z) &= \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \\
&= \mathbb{E}(\tau T + \delta_u U)Z] - \mathbb{E}[\tau T + \delta_u U]\mathbb{E}[Z]
\end{aligned}
$$

By linearity and exclusion restriction

$$
\boxed{\hat{\tau} = \frac{\hat{\mathrm{Cov}}(Y, Z)}{\hat{\mathrm{Cov}}(T, Z)}}
$$



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&= \mathbb{E}(\tau T + \delta_u U)Z] - \mathbb{E}[\tau T + \delta_u U]\mathbb{E}[Z] \\
&= \tau \mathbb{E}[TZ] + \delta_u \mathbb{E}[UZ] - \tau \mathbb{E}[T]\mathbb{E}[Z] - \delta_u \mathbb{E}[U]\mathbb{E}[Z]
\end{aligned}$$

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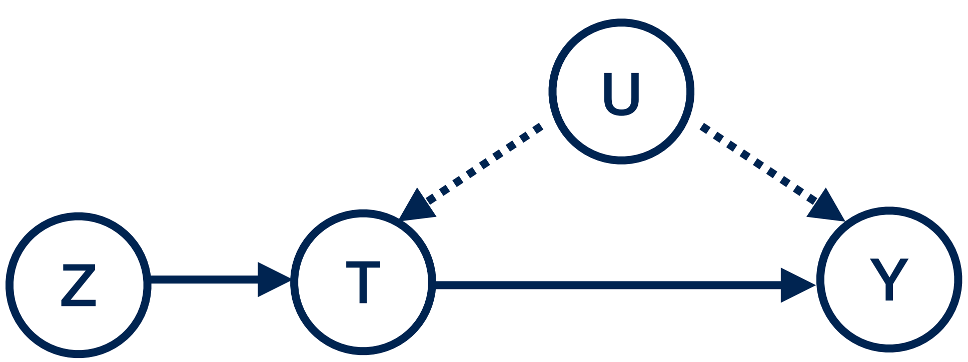

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&= \tau\mathbb{E}[TZ] + \delta_u\mathbb{E}[UZ] - \tau\mathbb{E}[T]\mathbb{E}[Z] - \delta_u\mathbb{E}[U]\mathbb{E}[Z] \\
&= \tau\mathrm{Cov}(T, Z) + \delta_U\mathrm{Cov}(U, Z)
\end{aligned}
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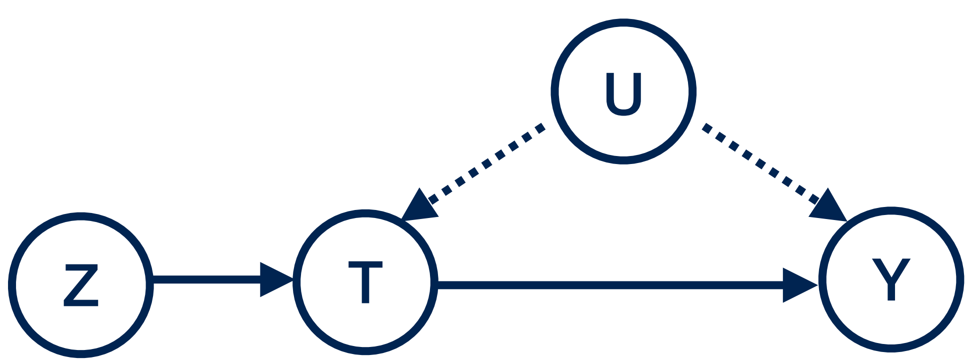

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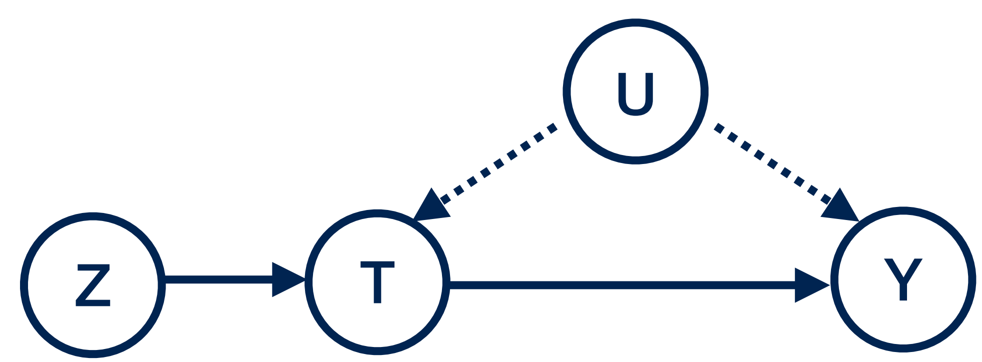
&= \tau \mathbb{E}[TZ] + \delta_u \mathbb{E}[UZ] - \tau \mathbb{E}[T]\mathbb{E}[Z] - \delta_u \mathbb{E}[U]\mathbb{E}[Z] \\
&= \tau \mathrm{Cov}(T, Z) + \delta_U \mathrm{Cov}(U, Z) \\
&= \tau \mathrm{Cov}(T, Z)
\end{aligned}
$$

Instrument is not confounded by U

$$\boxed{\hat{\tau} = \frac{\hat{\mathrm{Cov}}(Y, Z)}{\hat{\mathrm{Cov}}(T, Z)}}$$



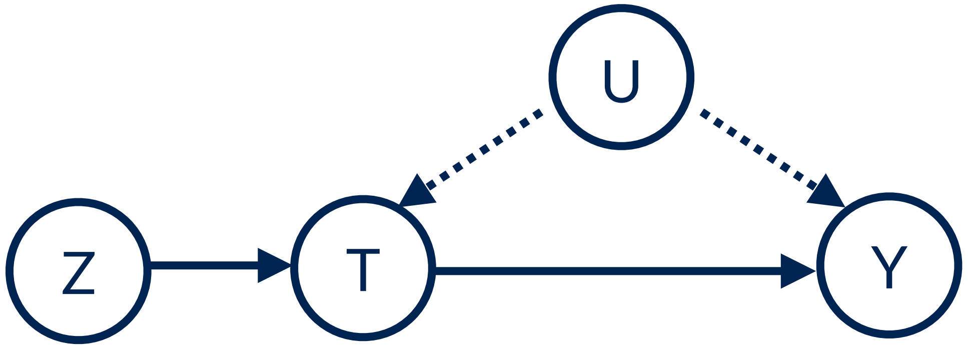
$$Y = \tau T + \delta_U U$$

IV Estimator: continuous variables case

Two-Stage Least Squares Estimator (linear regression):

1. Estimate $\mathbb{E}[T|Z]$, to obtain $\hat{T}$ in subspace

2. Estimate $\mathbb{E}[Y|\hat{T}]$, to obtain $\hat{\tau}$, which is the fitted coefficient in front of $\hat{T}$ in this regression.



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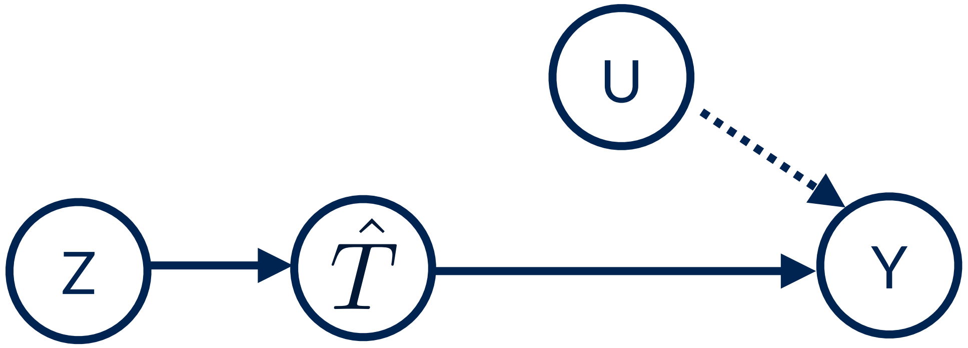

$$Y = \tau T + \delta_U U$$

Other remarks

Double-blind studies:
To ensure exclusion restriction, investigators withhold knowledge of the assigned treatment Z from participants and doctors

Example: Those randomly assigned z=1, receive aspirin, but those assigned z=0 receive placebo, do not. The pills look identical. Neither doctor nor patient knowns which is which, “double-blind placebo-controlled” randomised experiment.

Often not feasible, e.g. heart surgery, has no convincing placebo!

Overview of the course

- **Lecture 1**: Introduction & Motivation, why do we care about causality? Why deriving causality from observational data is non-trivial.
- **Lecture 2:** Recap of probability theory, variables, events, conditional probabilities, independence, law of total probability, Bayes' rule
- **Lecture 3**: Recap of regression, multiple regression, graphs, SCM
- **Lecture 4-20:**