These are exercises for self-study and exam preparation. All material is examinable unless otherwise mentioned.

Exercise 1. ***The chain rule factorisation***

Let (x, y, z) be three discrete random variables, each taking values in $\{0, 1\}$. Suppose their joint probability mass function is

x	y	z	$p(x, y, z)$
0	0	0	$\frac{1}{20}$
0	0	1	$\frac{2}{20}$
0	1	0	$\frac{3}{20}$
0	1	1	$\frac{4}{20}$
1	0	0	$\frac{1}{20}$
1	0	1	$\frac{3}{20}$
1	1	0	$\frac{2}{20}$
1	1	1	$\frac{4}{20}$

(a) *The chain rule for the ordering x, y, z is*

$$p(x, y, z) = p(x)p(y|x)p(z|x, y) \tag{1}$$

Determine the numerical values for all (conditional) pmfs in the above factorisation.

Solution. This requires repeat application of the sum and product rule. We start with computing $p(x)$:

$$p(x = 0) = \sum_{y,z} p(0, y, z) \tag{S.1}$$

$$= \frac{1 + 2 + 3 + 4}{20} = \frac{10}{20} = \frac{1}{2}. \tag{S.2}$$

Furthermore: $p(x = 1) = 1 - p(x = 0) = 1/2$.
Since

$$p(y|x) = \frac{p(x, y)}{p(x)} \tag{S.3}$$

we next compute $p(x, y)$ by marginalising out z from $p(x, y, z)$:

x	y	$p(x, y) = \sum_z p(x, y, z)$
0	0	$\frac{1+2}{20} = \frac{3}{20}$
0	1	$\frac{3+4}{20} = \frac{7}{20}$
1	0	$\frac{1+3}{20} = \frac{4}{20}$
1	1	$\frac{2+4}{20} = \frac{6}{20}$

The conditional $p(y|x)$ thus is:

x	y	$p(y\|x) = p(x,y)/p(x)$
0	0	$\frac{3}{10}$
0	1	$\frac{7}{10}$
1	0	$\frac{4}{10} = \frac{2}{5}$
1	1	$\frac{6}{10} = \frac{3}{5}$

For the last factor, we use

$$p(z|x, y) = \frac{p(x, y, z)}{p(x, y)} \tag{S.4}$$

We thus normalise the joint $p(x, y, z)$ with the values of the bivariate marginal $p(x, y)$ computed above.

x	y	z	$p(z\|x, y)$
0	0	0	$\frac{1}{3}$
0	0	1	$\frac{2}{3}$
0	1	0	$\frac{3}{7}$
0	1	1	$\frac{4}{7}$
1	0	0	$\frac{1}{4}$
1	0	1	$\frac{3}{4}$
1	1	0	$\frac{2}{6}$
1	1	1	$\frac{4}{6}$

(b) *Specify the chain-rule factorisation for the ordering z, y, x.*

Solution. We have

$$p(x, y, z) = p(z)p(y|z)p(x|y, z) \tag{S.5}$$

(c) *Compute the numerical values for each factor found in the factorisation above.*

Solution. We start with $p(z)$:

$$p(z = 0) = \sum_{x,y} p(x, y, z = 0) \tag{S.6}$$

$$\frac{1 + 3 + 1 + 2}{20} = \frac{7}{20} \tag{S.7}$$

Furthermore: $p(z = 1) = 1 - p(z = 0) = \frac{13}{20}$.
Since

$$p(y|z) = \frac{p(y, z)}{p(z)} \tag{S.8}$$

we next compute $p(y, z)$ by marginalising out x from $p(x, y, z)$

y	z	$p(y,z) = \sum_x p(x,y,z)$
0	0	$\frac{1+1}{20} = \frac{2}{20} = \frac{1}{10}$
0	1	$\frac{2+3}{20} = \frac{5}{20} = \frac{1}{4}$
1	0	$\frac{3+2}{20} = \frac{5}{20} = \frac{1}{4}$
1	1	$\frac{4+4}{20} = \frac{8}{20} = \frac{2}{5}$

The conditional $p(y|z)$ thus is:

y	z	$p(y|z) = p(y,z)/p(z)$
0	0	$\frac{2}{7}$
0	1	$\frac{5}{13}$
1	0	$\frac{5}{7}$
1	1	$\frac{8}{13}$

For $p(x|y,z)$, we use

$$p(x|y,z) = \frac{p(x,y,z)}{p(y,z)} \tag{S.9}$$

and thus normalise the joint $p(x,y,z)$ with the values of the bivariate marginal $p(y,z)$ computed above.

x	y	z	$p(x|y,z)$
0	0	0	$\frac{1}{2}$
0	0	1	$\frac{2}{5}$
0	1	0	$\frac{3}{5}$
0	1	1	$\frac{4}{8} = \frac{1}{2}$
1	0	0	$\frac{1}{2}$
1	0	1	$\frac{3}{5}$
1	1	0	$\frac{2}{5}$
1	1	1	$\frac{4}{8} = \frac{1}{2}$

(d) *Check that the pmf for $(x,y,z) = (1,1,0)$ is the same when computed with the chain-rule for the ordering x,y,z or z,y,x.*

Solution. For the ordering x,y,z, we have

$$\begin{aligned}
p(x=1,y=1,z=0) &= p(x=1)p(y=1|x=1)p(z=0|x=1,y=1) && \text{(S.10)}\\
&= \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{2}{6} && \text{(S.11)}\\
&= \frac{1}{10} && \text{(S.12)}
\end{aligned}$$

For the ordering z, y, x, we have

$$p(x=1, y=1, z=0) = p(z=0)p(y=1|z=0)p(x=1|y=1, z=0) \tag{S.13}$$
$$= \frac{7}{20} \cdot \frac{5}{7} \cdot \frac{2}{5} \tag{S.14}$$
$$= \frac{2}{20} = \frac{1}{10} \tag{S.15}$$

Exercise 2. ***Expectations***

Let (x, y, z) be three discrete random variables, each taking values in $\{0, 1\}$. Suppose their joint probability mass function is

x	y	z	$p(x, y, z)$
0	*0*	*0*	$\frac{1}{20}$
0	*0*	*1*	$\frac{2}{20}$
0	*1*	*0*	$\frac{3}{20}$
0	*1*	*1*	$\frac{4}{20}$
1	*0*	*0*	$\frac{1}{20}$
1	*0*	*1*	$\frac{3}{20}$
1	*1*	*0*	$\frac{2}{20}$
1	*1*	*1*	$\frac{4}{20}$

(a) Compute $\mathbb{E}[(x+1)^2]$

Solution. Because the expectation only involves x, we only need to know $p(x)$, which we can compute via the sum-rule, marginalising out y and z. We have

$$p(x=0) = \sum_{y,z} p(0, y, z) \tag{S.16}$$
$$= \frac{1+2+3+4}{20} = \frac{10}{20} = \frac{1}{2}. \tag{S.17}$$

Furthermore: $p(x=1) = 1 - p(x=0) = 1/2$.
We thus obtain

$$\mathbb{E}[(x+1)^2] = \sum_x p(x)(x+1)^2 \tag{S.18}$$
$$= \frac{1}{2}\left(1^2 + 2^2\right) \tag{S.19}$$
$$= \frac{5}{2} \tag{S.20}$$

(b) Compute $\mathbb{E}[(x+1)^2(y+2)]$

Solution. Because the expectation depends on a nonlinear combination of x and y, we need to know $p(x, y)$, which we can compute via the sum-rule, marginalising out z. We have

x	y	$p(x,y) = \sum_z p(x,y,z)$
0	0	$\frac{1+2}{20} = \frac{3}{20}$
0	1	$\frac{3+4}{20} = \frac{7}{20}$
1	0	$\frac{1+3}{20} = \frac{4}{20}$
1	1	$\frac{2+4}{20} = \frac{6}{20}$

We thus obtain

$$
\begin{aligned}
\mathbb{E}[(x+1)^2(y+2)] &= \sum_{x,y} p(x,y)(x+1)^2(y+2) && \text{(S.21)}\\
&= \frac{3}{20} \cdot 1^2 \cdot 2 + \frac{7}{20} \cdot 1^2 \cdot 3 + && \text{(S.22)}\\
&\quad \frac{4}{20} \cdot 2^2 \cdot 2 + \frac{6}{20} \cdot 2^2 \cdot 3 && \text{(S.23)}\\
&= \frac{6}{20} + \frac{21}{20} + \frac{32}{20} + \frac{72}{20} && \text{(S.24)}\\
&= \frac{131}{20} && \text{(S.25)}
\end{aligned}
$$

(c) *Compute* $\mathbb{E}[(x+1)^2(y+2)|x=0]$

Solution. Following the definition for conditional expectations, we have

$$
\begin{aligned}
\mathbb{E}[(x+1)^2(y+2)|x=0] &= \sum_{y} p(y|x=0)1^2(y+2) && \text{(S.26)}\\
&= p(y=0|x=0) \cdot 2 + p(y=1|x=0) \cdot 3 && \text{(S.27)}
\end{aligned}
$$

The conditional $p(y|x=0)$ is

$$
\begin{aligned}
p(y|x=0) &= \frac{p(x=0,y)}{p(x=0)} && \text{(S.28)}\\
&= \begin{cases} \frac{3}{10} & y=0 \\ \frac{7}{10} & y=1 \end{cases} && \text{(S.29)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}[(x+1)^2(y+2)|x=0] &= \frac{3}{10} \cdot 2 + \frac{7}{10} \cdot 3 && \text{(S.30)}\\
&= \frac{6+21}{10} && \text{(S.31)}\\
&= \frac{27}{10} && \text{(S.32)}
\end{aligned}
$$