Exercise 1. ***Inference for the binary symmetric channel***

Let x_0 be a binary random variable taking on states 0 and 1 with probability 1/2. A binary symmetric channel (BSC) flips an input bit with probability f, and leaves it unflipped with probability $1-f$. Let x_1 be the result of passing x_0 through the BSC. Hence we have that $p(x_1 = 0|x_0 = 0) = p(x_1 = 1|x_0 = 1) = 1 - f$. Now suppose that x_1 is passed through another BSC (also with flip probability f) to yield x_2. The graphical model is thus $x_0 \to x_1 \to x_2$.

(a) You observe that $x_2 = 1$. Compute $p(x_0 = 1|x_2 = 1)$.

Solution. There are many ways to solve this question. We start from Bayes' theorem

$$p(x_0 = 1|x_2 = 1) = \frac{p(x_0 = 1, x_2 = 1)}{p(x_2 = 1)} = \frac{\sum_{x_1=0,\,1} p(x_0 = 1, x_1, x_2 = 1)}{p(x_2 = 1)}. \tag{S.1}$$

From the factorization of the grapical model $x_0 \to x_1 \to x_2$ we have

$$p(x_0 = 1, x_1, x_2 = 1) = p(x_0 = 1)p(x_1|x_0 = 1)p(x_2 = 1|x_1). \tag{S.2}$$

Hence we have that

$$p(x_0 = 1, x_2 = 1) = \sum_{x_1 \in 0,1} p(x_0 = 1)p(x_1|x_0 = 1)p(x_2 = 1|x_1) = \frac{1}{2}f^2 + \frac{1}{2}(1-f)^2, \tag{S.3}$$

corresponding to no flips or two flips. See Fig. 1 for an illustration.

To get the denominator $p(x_2 = 1)$, we have that

$$p(x_2 = 1) = \sum_{x_0,\, x_1 \in 0,1} p(x_0)p(x_1|x_0)p(x_2 = 1|x_1) = \frac{1}{2}f^2 + \frac{1}{2}(1-f)^2 + 2\frac{1}{2}f(1-f) = \frac{1}{2}, \tag{1}$$

which might have been expected anyway by symmetry. Thus

$$p(x_0 = 1|x_2 = 1) = \frac{p(x_0 = 1, x_2 = 1)}{p(x_2 = 1)} = 2 \cdot \left[\frac{1}{2}f^2 + \frac{1}{2}(1-f)^2\right] = f^2 + (1-f)^2. \tag{2}$$

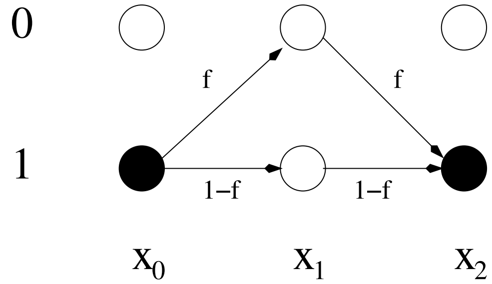

Figure 1: Lattice diagram with showing the variables to x_0, x_1, x_2 taking on the values 0 and 1. The paths shown start at $x_0 = 1$ and end at $x_2 = 1$ (shown shaded).

