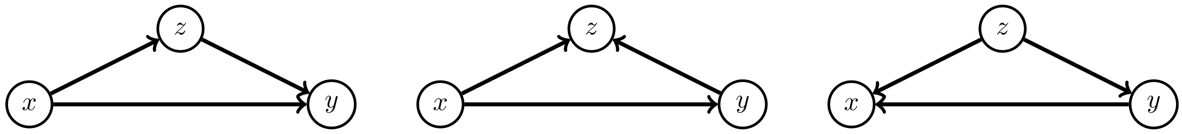
These are exercises for self-study and exam preparation. All material is examinable unless otherwise mentioned.

Exercise 1. *Computing postinterventional distributions*

Consider the following causal DAGs for three discrete-valued random variables x, y, z:



(a) (b) (c)

(a) *Compute $p(y; do(x) = a)$ for DAG (a). Express the result in terms of the conditional probability distributions $p(x_i|\text{pa}_i)$ of the graphical model defined the DAG.*

Solution. The DAG modelling the intervention on x is the same as the original graph since x is a root node. Atomic interventions correspond to using $p'(x) = \delta(x - a)$ as interventional distribution, with

$$\delta(x - a) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \tag{S.1}$$

From the graph, we thus obtain the following factorisation

$$p(x, y, z; do(x) = a) = \delta(x - a)p(z|x)p(y|z, x). \tag{S.2}$$

To obtain $p(y; do(x) = a)$ we marginalise out x and z, which gives

$$p(y; do(x) = a) = \sum_{x,z} p(x, y, z; do(x) = a) \tag{S.3}$$

$$= \sum_{x,z} \delta(x - a)p(z|x)p(y|z, x) \tag{S.4}$$

$$= \sum_{z} p(z|x = a)p(y|z, x = a) \tag{S.5}$$

This cannot be simplified any further and is the desired expression for $p(y; do(x) = a)$. A variable like z in the graph, being on a directed path from cause x to effect y, is called a mediator variable.

(b) *Compute $p(y; do(x) = a)$ for DAG (b). Express the result in terms of the conditional probability distributions $p(x_i|\text{pa}_i)$ of the graphical model defined the DAG.*

Solution. The DAG modelling the intervention on x is the same as the original graph since x is a root node. From the graph, we can write down the factorisation

$$p(x, y, z; do(x) = a) = \delta(x - a)p(y|x)p(z|x, y) \tag{S.6}$$

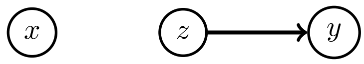
To obtain $p(y; do(x) = a)$ we marginalise out x and z, which gives

$$
\begin{aligned}
p(y; do(x) = a) &= \sum_{x,z} p(x, y, z; do(x) = a) && \text{(S.7)} \\
&= \sum_{x,z} \delta(x - a) p(y|x) p(z|x, y) && \text{(S.8)} \\
&= \sum_{z} p(y|x = a) p(z|x = a, y) && \text{(S.9)} \\
&= p(y|x = a) \sum_{z} p(z|x = a, y) && \text{(S.10)} \\
&= p(y|x = a) && \text{(S.11)}
\end{aligned}
$$

This result makes intuitive sense since z does not causally affect y, and hence any change of its distribution due to the intervention on x does not propagate to y. Variable x only causally affects y via the direct $x \to y$ effect and hence $p(y; do(x) = a) = p(y|x = a)$.

(c) *Compute $p(y; do(x) = a)$ for DAG (c). Express the result in terms of the conditional probability distributions $p(x_i|\mathrm{pa}_i)$ of the graphical model defined the DAG.*

Solution. The DAG modelling the intervention on x is obtained by removing all incoming arrows into x, which gives



The joint distribution over it factorises as

$$
p(x, y, z; do(x) = a) = \delta(x - a) p(z) p(y|z) \tag{S.12}
$$

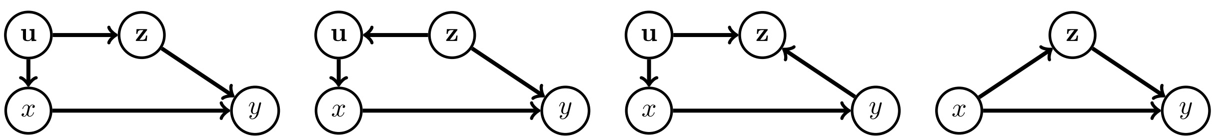
To obtain $p(y; do(x) = a)$ we marginalise out x and z, which gives

$$
\begin{aligned}
p(y; do(x) = a) &= \sum_{x,z} p(x, y, z; do(x) = a) && \text{(S.13)} \\
&= \sum_{x,z} \delta(x - a) p(z) p(y|z) && \text{(S.14)} \\
&= p(y) && \text{(S.15)}
\end{aligned}
$$

Given that x is a leaf variable in the original DAG, intervening on it does not change any of the upstream distributions. Hence, we have that $p(y; do(x)) = p(y)$.

Exercise 2. *Backdoor adjustment*

For each of the following DAGs G, explain whether $\mathbf{z}$ *can be used to compute $p(y; do(x))$ via backdoor adjustment.*



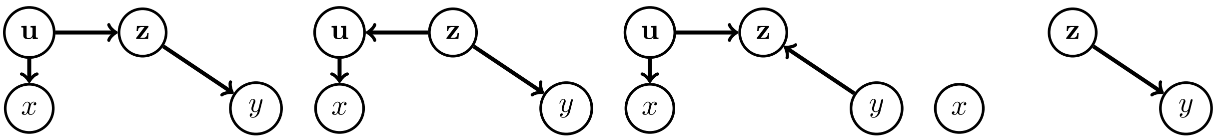
(a) (b) (c) (d)

Solution. The variable $\mathbf{z}$ must satisfy the following two criteria for $p(y; do(x) = a) = \mathbb{E}_{p(\mathbf{z})}\left[p(y|x = a, \mathbf{z})\right]$ to hold:

1. $x \perp\!\!\!\perp y|\mathbf{z}$ in $G_{\underline{x}}$, and
2. no component of $\mathbf{z}$ is a descendant of x,

where $G_{\underline{x}}$ denotes the graph where all outgoing arrows from x are removed.

The following graphs show $G_{\underline{x}}$



(a) (b) (c) (d)

We see that (a), (b), and (d) satisfy the first criterion: In (a) and (b) $\mathbf{z}$ blocks the $x - \mathbf{u} - \mathbf{z} - y$ trail since $\mathbf{z}$ is either in a head-tail or tail-tail configuration. For graph (c), $\mathbf{z}$ is in a collider configuration so that conditioning on it opens the trail from x to y and the independency does not hold.

From the original graphs G, we see that, for (a) and (b), $\mathbf{z}$ is a non-descendant of x, so the second criterion holds these graphs. For (d), however, $\mathbf{z}$ is a descendant of x so the second criterion does not hold.

In conclusion, the graphs (a) and (b), we can use $\mathbf{z}$ for the backdoor adjustment. However, not so for (c) and (d). In (c), by conditioning on $\mathbf{z}$, we would open a backdoor path while in (d), $\mathbf{z}$ is a mediator and hence part of a causal path between x and y.

Exercise 3. *Backdoor adjustment for non-atomic interventional distributions*

The backdoor adjustment criterion says that if $\mathbf{z}$ satisfies

1. $x_i \perp\!\!\!\perp x_k|\mathbf{z}$ *in* $G_{\underline{x_k}}$, *and*
2. *no component of* $\mathbf{z}$ *is a descendant of* x_k,

then $p(x_i; do(x_k) = a) = \mathbb{E}_{p(\mathbf{z})}\left[p(x_i|x_k = a, \mathbf{z})\right]$. *Here,* $G_{\underline{x_k}}$ *denotes the graph where all outgoing arrows from* x_k *are removed.*

Extend this result to $p(x_i; do(x_k) \sim p'(x_k))$ *where* $p'(x_k)$ *is a general interventional distribution. For simplicity, you can assume that the random variables are discrete-valued.*

Solution. We start with the general expression for the postinterventional distribution for a causal DAG:

$$p(\mathbf{x}; do(x_k) \sim p') = \prod_{i \neq k} p(x_i|\mathrm{pa}_i) \cdot p'(x_k) \tag{S.16}$$

Assume that x_k is discrete and that it can take on values in the set $\mathcal{X}$. We next note that $p'(x_k)$ can be expressed as

$$p'(x_k) = \sum_{a \in \mathcal{X}} \delta(x_k - a) p'(x_k = a) \tag{S.17}$$

where $\delta(x-a)=1$ if $x=a$ and is zero otherwise. Note that $p'(x_k=a)$ denotes the value of $p'(x_k)$ when x_k equals a; it is thus a function of a and not x_k.

Continuing with the discrete case, we thus have

$$
\begin{aligned}
p(\mathbf{x}; do(x_k) \sim p') &= \prod_{i \neq k} p(x_i|\mathrm{pa}_i) \cdot p'(x_k) && \text{(S.18)}\\
&= \prod_{i \neq k} p(x_i|\mathrm{pa}_i) \cdot \sum_{a \in \mathcal{X}} \delta(x_k - a) p(x_k = a) && \text{(S.19)}\\
&= \sum_{a \in \mathcal{X}} \left(\prod_{i \neq k} p(x_i|\mathrm{pa}_i) \delta(x_k - a) \right) p'(x_k = a) && \text{(S.20)}\\
&= \sum_{a \in \mathcal{X}} p(\mathbf{x}; do(x_k) = a) p'(x_k = a) && \text{(S.21)}
\end{aligned}
$$

Since this is an expectation over $p'(x_k)$, this means that $p(\mathbf{x}; do(x_k) \sim p')$ is obtained by first computing the postinterventional distribution for atomic interventions and then taking their expected value (weighted average). This is a general result that connects the effects of non-atomic interventions to atomic ones.

Going back to the original question, we note that $p(x_i; do(x_k) \sim p'(x_k))$ is obtained from $p(\mathbf{x}; do(x_k) \sim p'(x_k))$ by marginalising over all variables but x_i. Denoting them by $\mathbf{x}_{\setminus i}$, we thus obtain

$$
\begin{aligned}
p(x_i; do(x_k) \sim p') &= \sum_{\mathbf{x}_{\setminus i}} p(\mathbf{x}; do(x_k) \sim p') && \text{(S.22)}\\
&= \sum_{\mathbf{x}_{\setminus i}} \sum_{a \in \mathcal{X}} p(\mathbf{x}; do(x_k) = a) p'(x_k = a) && \text{(S.23)}\\
&= \sum_{a \in \mathcal{X}} \left(\sum_{\mathbf{x}_{\setminus i}} p(\mathbf{x}; do(x_k) = a) \right) p'(x_k = a) && \text{(S.24)}\\
&= \sum_{a \in \mathcal{X}} p(x_i; do(x_k) = a) p'(x_k = a) && \text{(S.25)}
\end{aligned}
$$

What remains to be done is inserting the expression for $p(x_i; do(x_k) = a)$ that we obtain from the backdoor criterion, which gives the desired result:

$$
\begin{aligned}
p(x_i; do(x_k) \sim p') &= \sum_{a \in \mathcal{X}} \mathbb{E}_{p(\mathbf{z})} \left[p(x_i|x_k = a, \mathbf{z}) \right] p'(x_k = a) && \text{(S.26)}\\
&= \mathbb{E}_{p'(x_k)} \mathbb{E}_{p(\mathbf{z})} \left[p(x_i|x_k, \mathbf{z}) \right] && \text{(S.27)}
\end{aligned}
$$

The criteria for $\mathbf{z}$ stay the same as for the atomic interventions. This generalises the formula for the adjustment for direct causes: $p(x_i; do(x_k) \sim p') = \mathbb{E}_{p'(x_k)p(\mathrm{pa}_k)} \left[p(x_i|x_k, \mathrm{pa}_k) \right]$.