These are exercises for self-study and exam preparation. All material is examinable unless otherwise mentioned.

Exercise 1. ***Optimal actions for different loss functions***

Assume that $h \in \{1, \dots, 5\}$ *with probabilities* $f(h)$ *equal to*

$$f(h=1) = 0.2, \quad f(h=2) = 0.3, \quad f(h=3) = 0, \quad f(h=4) = 0.1, \quad f(h=5) = 0.4 \tag{1}$$

Compute the action a *that minimises the expected loss* $\mathbb{E}_{f(h)}[\ell(h, a)]$ *when:*

(a) $\ell(h, a) = (h - a)^2$

Solution. The optimal action $a^*(f)$ for the quadratic loss is given by the mean $\mathbb{E}_{f(h)}[h]$, which equals

$$\begin{aligned} \mathbb{E}_{f(h)}[h] &= \sum_h hp(h) && \text{(S.1)} \\ &= 1 \cdot 0.2 + 2 \cdot 0.3 + 3 \cdot 0 + 4 \cdot 0.1 + 5 \cdot 0.4 && \text{(S.2)} \\ &= 3.2 && \text{(S.3)} \end{aligned}$$

(b) $\ell(h, a) = |h - a|$

Solution. The optimal action $a^*(f)$ for the absolute error loss is given by the median of f. To compute the median we compute the cumulative distribution function (cdf) $F(\alpha) = \mathbb{P}(h \leq \alpha)$. Even for discrete random variables, the cdf is defined for all values of α. For discrete h, the cdf thus will be piecewise constant with jump points at

$$F(1) = 0.2, \quad F(2) = 0.5, \quad F(4) = 0.6, \quad F(5) = 1 \tag{S.4}$$

The median is any number m that satisfies $F(m^-) \leq 0.5 \leq F(m)$ where m^- is the lower limit.

Since $F(\alpha)$ is equal to 0.5 on the interval $[2, 4)$ with a jump point at 4, the median is here not unique, any number in the interval $[2, 4]$ satisfies the definition to be a median. The endpoint 4 is included since $F(4^-) = 0.5 \leq 0.5$ and $F(4) = 0.6 \geq 0.5$. There are several approaches to deal with the non-uniqueness: One is to report the whole interval $[2, 4]$; if a single number is required, a popular approach is to report the smallest median, i.e. 2 here.

(c) $\ell(h, a) = 1 - \mathbb{1}(h = a)$

Solution. The optimal action $a^*(f)$ for the zero-one loss is given by the mode of f, i.e. $\text{argmax}_h\, f(h)$, which is here $h = 5$.

(d) $\ell(h, a) = -\log a(h)$ *where choosing an action* a *means choosing a distribution over* h*.*

Solution. The optimal action for the log-loss when the expectation is taken over f is f itself.

Exercise 2. ***Causal decision theory applied to the kidney stone example***

The table summarizes outcomes for two types of surgery ($T = a$ and $T = b$) to remove kidney stones. Success rates are presented both overall and by stone size. For each treatment and size category, the table also reports the number of successes ($R = 1$) and the number of patients treated. For example, 192/263 means that 263 patients received treatment a and for 192 of them the surgery was successful ($R = 1$).

	Overall success rate	*Small stones*	*Large stones*
Treatment a	*78% (273/350)*	*93% (81/87)*	*73% (192/263)*
Treatment b	*83% (289/350)*	*87% (234/270)*	*69% (55/80)*

To decide which treatment to choose when the size of the kidney stone is unknown, we specify the following loss function:

$$\ell_0(R) = \begin{cases} c & \text{if } R = 0 \\ 0 & \text{if } R = 1 \end{cases} \tag{2}$$

where $c > 0$. The loss ignores recovery time and other considerations, and simply assigns a penalty $c > 0$ to an unsuccessful surgery.

(a) Determine the treatment T that minimises

$$\mathcal{R}(T) = \mathbb{E}_{p(R;do(T))}\left[\ell_o(R)\right]. \tag{3}$$

You may use that

$$p(R = 1; do(T) = a) = 0.833 \qquad p(R = 1; do(T) = b) = 0.779 \tag{4}$$

Solution. The loss equals $\mathcal{R}(T) = c \cdot p(R = 0; do(T))$. Hence we choose the treatment for which the probability of failure is smallest, or, in other words, the treatment for which the probability $p(R = 1; do(T))$ is largest. This is the case for $T = a$.

(b) Assume now that the success rates are considered uncertain (e.g because they are computed from a small number of patients only). How can this uncertainty be incorporated in the decision making?

Solution. We can consider the success rates in the table to be random variables $\mathbf{h}$. Their distribution is not affected by the action that we are taking now, hence $p(\mathbf{h}; do(T)) = p(\mathbf{h})$. As seen in the lecture, we can then express the risk as

$$\mathcal{R}(T) = \mathbb{E}_{p(\mathbf{h})}\mathbb{E}_{p(R|\mathbf{h};do(T))}\left[\ell_o(R)\right]. \tag{S.5}$$

$$= c\mathbb{E}_{p(\mathbf{h})}\left[p(R = 0|\mathbf{h}; do(T))\right]. \tag{S.6}$$

This has the form of an expected loss with $\ell(\mathbf{h}, T) = p(R = 0|\mathbf{h}; do(T))$, the probability of failure when taking action T for a given $\mathbf{h}$. Hence, for each value of $\mathbf{h}$, we compute $p(R = 0|\mathbf{h}; do(T))$ and then choose the treatment T for which the expected probability of failure is smallest.

Exercise 3. ***Classification with an asymmetric loss***

We here derive the optimal policy for binary classification when the costs of false positives and false negatives are unequal. Denote by $h \in \{0, 1\}$ the true class label and the predicted label by $a \in \{0, 1\}$. We consider the following loss

h	a	$\ell(h, a)$
0	*0*	*0*
1	*0*	c_{fn}
0	*1*	c_{fp}
1	*1*	*0*

where $c_{fn} > 0$ indicates the cost of a false negative and $c_{fp} > 0$ the cost of a false positive. We assume that we were given data $\mathbf{x}$ and that can compute the posterior $p(h|\mathbf{x})$. Derive the policy $a^*(\mathbf{x})$ that minimises the posterior expected loss, i.e.

$$a^*(\mathbf{x}) = \operatorname*{argmin}_a \mathbb{E}_{p(h|\mathbf{x})} \left[\ell(h, a)\right] \tag{5}$$

Solution. We compute the risk $\mathbb{E}_{p(h|\mathbf{x})}[\ell(h, a)]$ for the two actions $a = 0$ and $a = 1$:

$$\mathbb{E}_{p(h|\mathbf{x})} \left[\ell(h, a = 0)\right] = p(h = 1|\mathbf{x})c_{fn} \qquad \mathbb{E}_{p(h|\mathbf{x})} \left[\ell(h, a = 1)\right] = p(h = 0|\mathbf{x})c_{fp} \tag{S.7}$$

The decision rule thus becomes $a^*(\mathbf{x}) = 1$ if $p(h = 0|\mathbf{x})c_{fp} < p(h = 1|\mathbf{x})c_{fn}$, and $a^*(\mathbf{x}) = 0$ if the inequality is reversed (if we have a tie, we may take a random decision). That is, we choose label $a = 1$ if

$$p(h = 0|\mathbf{x})c_{fp} < p(h = 1|\mathbf{x})c_{fn} \tag{S.8}$$

$$\frac{c_{fp}}{c_{fn}} < \frac{p(h = 1|\mathbf{x})}{p(h = 0|\mathbf{x})} \tag{S.9}$$

Denoting c_{fp}/c_{fn} by λ and using that $p(h = 0|\mathbf{x}) = 1 - p(h = 1|\mathbf{x})$ we can manipulate the above equation as follows:

$$\lambda < \frac{p(h = 1|\mathbf{x})}{1 - p(h = 1|\mathbf{x})} \tag{S.10}$$

$$\lambda - \lambda p(h = 1|\mathbf{x}) < p(h = 1|\mathbf{x}) \tag{S.11}$$

$$\lambda < p(h = 1|\mathbf{x})(1 + \lambda) \tag{S.12}$$

$$\frac{\lambda}{1 + \lambda} < p(h = 1|\mathbf{x}) \tag{S.13}$$

We thus see that as the relative cost λ of a false positive increases, the evidence in favour of $h = 1$ must become stronger, i.e. the posterior $p(h = 1|\mathbf{x})$ must become closer to one for the optimal decision to be $a = 1$.

Exercise 4. ***Penalised squared loss***

We consider the squared loss subject to a squared penalty for the action to deviate from zero,

$$\ell(h, a) = (h - a)^2 + \lambda a^2, \tag{6}$$

where $\lambda > 0$ is a weighting factor.

Derive the action that minimise the expected loss $\mathbb{E}_{f(h)} \left[\ell(h, a)\right]$.

Solution. Let $m = \mathbb{E}_{f(h)}[h]$. Since

$$\begin{aligned}
\mathbb{E}_{f(h)}\left[(h-a)^2\right] &= \mathbb{E}_{f(h)}\left[(h-m+m-a)^2\right] && \text{(S.14)}\\
&= \mathbb{E}_{f(h)}\left[(h-m)^2 + 2(h-m)(m-a) + \right. \\
&\quad \left. + (m-a)^2\right] && \text{(S.15)}\\
&= \mathbb{E}_{f(h)}\left[(h-m)^2\right] + 0 + (m-a)^2 && \text{(S.16)}\\
&= \mathbb{V}(h) + (m-a)^2 && \text{(S.17)}
\end{aligned}$$

we have

$$\mathbb{E}_{f(h)}\left[\ell(h,a)\right] = \mathbb{V}(h) + (m-a)^2 + \lambda a^2 \tag{S.18}$$

The derivative with respect to a is

$$-2(m-a) + 2\lambda a = 2a(1+\lambda) - 2m. \tag{S.19}$$

Setting the derivative to zero and solving for a gives

$$a = \frac{m}{1+\lambda}. \tag{S.20}$$

Since the second derivative of the expected loss is positive, the above is a minimum. Hence the optimal action is

$$a^* = \frac{1}{1+\lambda}\mathbb{E}_{f(h)}[h]. \tag{S.21}$$

The role of the penalty term is to shrink the mean $\mathbb{E}_{f(h)}[h]$ to zero.