These are exercises for self-study and exam preparation. All material is examinable unless otherwise mentioned.

Exercise 1. ***Minimal I-maps***

(a) Assume that the graph G in Figure 1 is a perfect I-map for $p(a, z, q, e, h)$. Determine the minimal directed I-map using the ordering (e, h, q, z, a). Is the obtained graph I-equivalent to G?

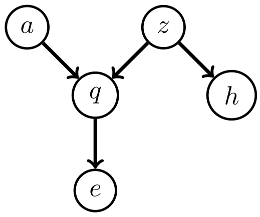


Figure 1: Perfect I-map G for Exercise 1, question (a).

(b) For the collection of random variables (a, z, h, q, e) you are given the following Markov blankets for each variable:

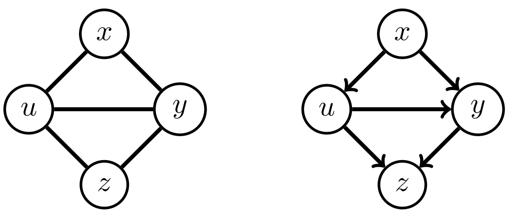
- MB(a) = {q,z}
- MB(z) = {a,q,h}
- MB(h) = {z}
- MB(q) = {a,z,e}
- MB(e) = {q}

(i) Draw the undirected minimal I-map representing the independencies.

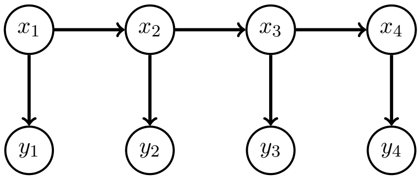
(ii) Indicate a Gibbs distribution that satisfies the independence relations specified by the Markov blankets.

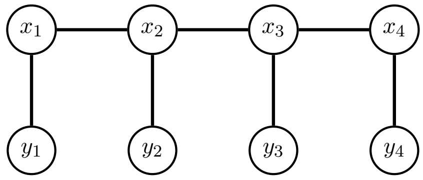
Exercise 2. ***I-equivalence between directed and undirected graphs***

(a) Verify that the following two graphs are I-equivalent by listing and comparing the independencies that each graph implies.

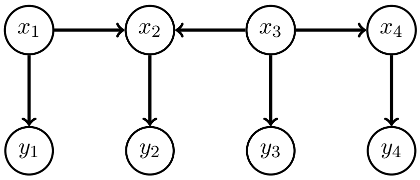


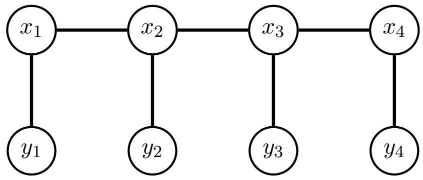
(b) Are the following two graphs, which are directed and undirected hidden Markov models, I-equivalent?





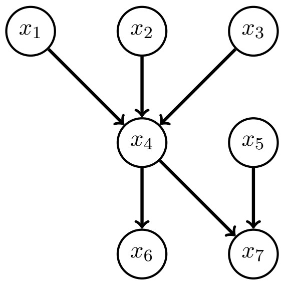
(c) Are the following two graphs I-equivalent?





Exercise 3. ***Moralisation exercise***

For the DAG G below find the minimal undirected I-map for $\mathcal{I}(G)$.



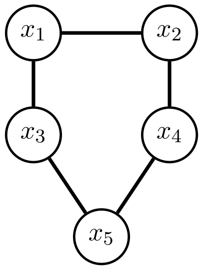
Exercise 4. ***Triangulation: Converting undirected graphs to directed minimal I-maps***

In the lecture, we have seen a recipe for constructing directed minimal I-maps for $\mathcal{I}(p)$. We here adapt it to build a directed minimal I-map for $\mathcal{I}(H)$, where H is an undirected graph. The difference to the procedure in the lecture is that we here use the graph H to determine independencies rather than the distribution p.

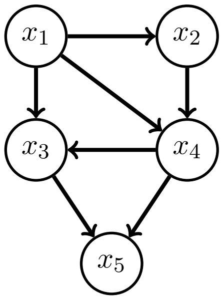
1. Choose an ordering of the random variables.
2. For all variables x_i, use H to determine a *minimal* subset π_i of the predecessors pre_i such that
$$x_i \perp\!\!\!\perp (\text{pre}_i \setminus \pi_i) \mid \pi_i$$
holds.
3. Construct a DAG with the π_i as parents pa_i of x_i.

Remarks: (1) Directed minimal I-maps obtained with different orderings are generally not I-equivalent. (2) The directed minimal I-maps obtained with the above method are always chordal graphs. Chordal graphs are graphs where the longest trail without shortcuts is a triangle (https://en.wikipedia.org/wiki/Chordal_graph). They are thus also called triangulated graphs. We obtain chordal graphs because if we had trails without shortcuts that involved more than 3 nodes, we would necessarily have an immorality in the graph. But immoralities encode independencies that an undirected graph cannot represent, which would make the DAG not an I-map for $\mathcal{I}(H)$ any more.

(a) Let H be the undirected graph below. Determine the directed minimal I-map for $\mathcal{I}(H)$ with the variable ordering x_1, x_2, x_3, x_4, x_5.



(b) For the undirected graph from question (a) above, which variable ordering yields the directed minimal I-map below?

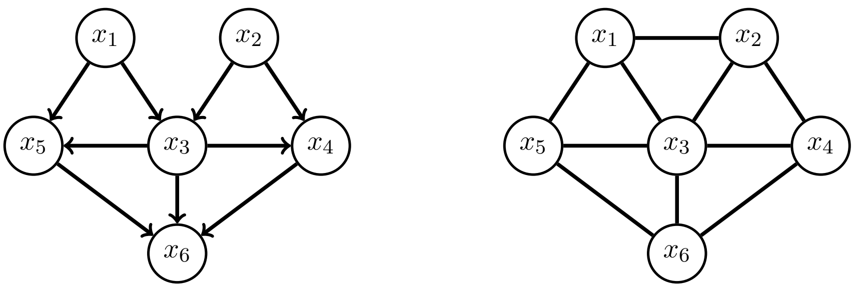


Exercise 5. ***I-maps, minimal I-maps, and I-equivalency***

Consider the following probability density function for random variables $x_1, \ldots, x_6$.

$$p_a(x_1, \ldots, x_6) = p(x_1)p(x_2)p(x_3|x_1, x_2)p(x_4|x_2)p(x_5|x_1)p(x_6|x_3, x_4, x_5)$$

For each of the two graphs below, explain whether it is a minimal I-map, not a minimal I-map but still an I-map, or not an I-map for the independencies that hold for p_a.



graph 1 graph 2