Exercise 1. ***Factorisation and independencies for undirected graphical models***

Consider the undirected graphical model defined by the graph in Figure 1.

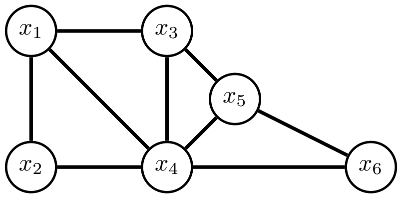


Figure 1: Graph for Exercise 1

(a) *What is the set of Gibbs distributions that is induced by the graph?*

Solution. The graph in Figure 1 has four maximal cliques:

$$(x_1, x_2, x_4) \quad (x_1, x_3, x_4) \quad (x_3, x_4, x_5) \quad (x_4, x_5, x_6)$$

The Gibbs distributions are thus

$$p(x_1, \ldots, x_6) \propto \phi_1(x_1, x_2, x_4)\phi_2(x_1, x_3, x_4)\phi_3(x_3, x_4, x_5)\phi_4(x_4, x_5, x_6)$$

(b) *Let p be a pdf that factorises according to the graph. Does $p(x_3|x_2, x_4) = p(x_3|x_4)$ hold?*

Solution. $p(x_3|x_2, x_4) = p(x_3|x_4)$ means that $x_3 \perp\!\!\!\perp x_2 \mid x_4$. We can use the graph to check whether this generally holds for pdfs that factorise according to the graph. There are multiple trails from x_3 to x_2, including the trail (x_3, x_1, x_2), which is not blocked by x_4. From the graph, we thus cannot conclude that $x_3 \perp\!\!\!\perp x_2 \mid x_4$, and $p(x_3|x_2, x_4) = p(x_3|x_4)$ will generally not hold (the relation may hold for some carefully defined factors ϕ_i).

(c) *Explain why $x_2 \perp\!\!\!\perp x_5 \mid x_1, x_3, x_4, x_6$ holds for all distributions that factorise over the graph.*

Solution. Distributions that factorise over the graph satisfy the pairwise Markov property. Since x_2 and x_5 are not neighbours, and x_1, x_3, x_4, x_6 are the remaining nodes in the graph, the independence relation follows from the pairwise Markov property.

(d) *Assume you would like to approximate $\mathbb{E}(x_1x_2x_5 \mid x_3, x_4)$, i.e. the expected value of the product of x_1, x_2, and x_5 given x_3 and x_4, with a sample average. Do you need to have joint observations for all five variables $x_1, \ldots, x_5$?*

Solution. In the graph, all trails from $\{x_1, x_2\}$ to x_5 are blocked by $\{x_3, x_4\}$, so that $x_1, x_2 \perp\!\!\!\perp x_5 \mid x_3, x_4$. We thus have

$$\mathbb{E}(x_1x_2x_5 \mid x_3, x_4) = \mathbb{E}(x_1x_2 \mid x_3, x_4)\mathbb{E}(x_5 \mid x_3, x_4).$$

Hence, we only need joint observations of (x_1, x_2, x_3, x_4) and (x_3, x_4, x_5). Variables (x_1, x_2) and x_5 do not need to be jointly measured.

Exercise 2. ***Factorisation from the Markov blankets***

Assume you know the following Markov blankets for all variables $x_1, \ldots, x_4, y_1, \ldots y_4$ of a pdf or pmf $p(x_1, \ldots, x_4, y_1, \ldots, y_4)$.

$$MB(x_1) = \{x_2, y_1\} \qquad MB(x_2) = \{x_1, x_3, y_2\} \qquad MB(x_3) = \{x_2, x_4, y_3\} \qquad MB(x_4) = \{x_3, y_4\} \tag{1}$$

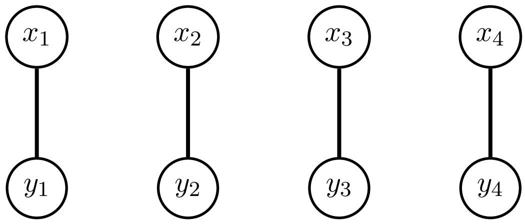
$$MB(y_1) = \{x_1\} \qquad MB(y_2) = \{x_2\} \qquad MB(y_3) = \{x_3\} \qquad MB(y_4) = \{x_4\} \tag{2}$$

Assuming that p is positive for all possible values of its variables, how does p factorise?

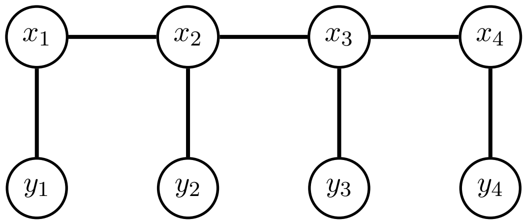
Solution. In undirected graphical models, the Markov blanket for a variable is the same as the set of its neighbours. Hence, when we are given all Markov blankets we know what local Markov property p must satisfy. For positive distributions we have an equivalence between p satisfying the local Markov property and p factorising over the graph. Hence, to specify the factorisation of p it suffices to construct the undirected graph H based on the Markov blankets and then read out the factorisation.

We need to build a graph where the neighbours of each variable equals the indicated Markov blanket. This can be easily done by starting with an empty graph and connecting each variable to the variables in its Markov blanket.

We see that each y_i is only connected to x_i. Including those Markov blankets we get the following graph:



Connecting the x_i to their neighbours according to the Markov blanket thus gives:



The graph has maximal cliques of size two, namely the $x_i - y_i$ for $i = 1, \ldots, 4$, and the $x_i - x_{i+1}$ for $i = 1, \ldots, 3$. Given the equivalence between the local Markov property and factorisation for positive distributions, we know that p must factorise as

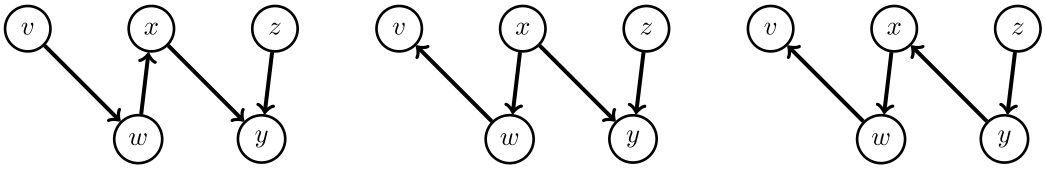
$$p(x_1, \ldots, x_4, y_1, \ldots, y_4) = \frac{1}{Z} \prod_{i=1}^{3} m_i(x_i, x_{i+1}) \prod_{i=1}^{4} g_i(x_i, y_i), \tag{S.1}$$

where $m_i(x_i, x_{i+1}) > 0$, $g(x_i, y_i) > 0$ are positive factors (potential functions).

The graphical model corresponds to an undirected version of a hidden Markov model where the x_i are the unobserved (latent, hidden) variables and the y_i are the observed ones. Note that the x_i form a Markov chain.

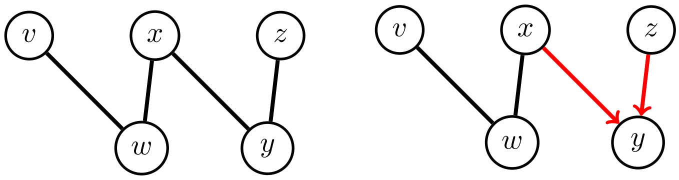
Exercise 3. ***I-equivalence***

(a) *Which of three graphs represent the same set of independencies? Explain.*



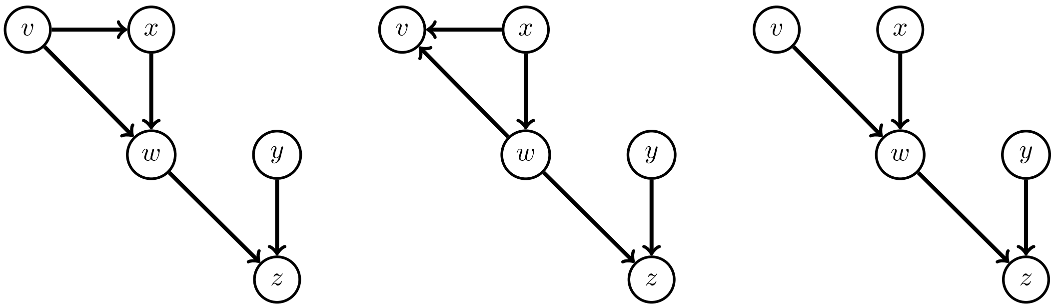
Graph 1 *Graph 2* *Graph 3*

Solution. To check whether the graphs are I-equivalent, we have to check the skeletons and the immoralities. All have the same skeleton, but graph 1 and graph 2 also have the same immorality. The answer is thus: graph 1 and 2 encode the same independencies.



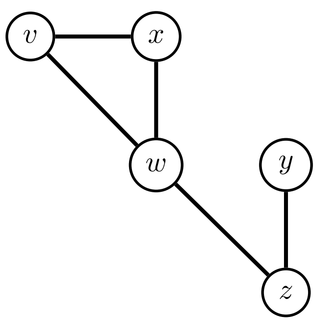
skeleton immorality

(b) *Which of three graphs represent the same set of independencies? Explain.*

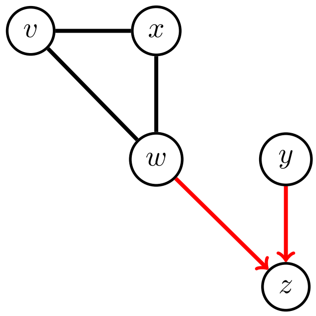


Graph 1 *Graph 2* *Graph 3*

Solution. The skeleton of graph 3 is different from the skeleton of graphs 1 and 2, so that graph 3 cannot be I-equivalent to graph 1 or 2, and we do not need to further check the immoralities for graph 3. Graph 1 and 2 have the same skeleton, and they also have the same immorality. Hence, graph 1 and 2 are I-equivalent. Note that node w in graph 1 is in a collider configuration along trail $v - w - x$ but it is not an immorality because its parents are connected (covering edge); equivalently for node v in graph 2.

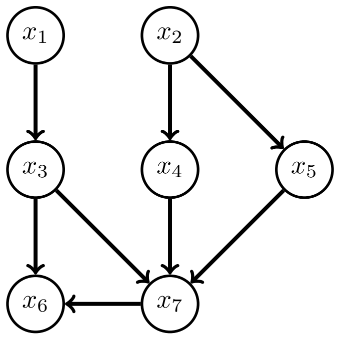


skeleton



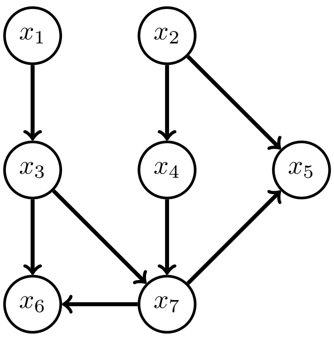
immorality

(c) *Assume the graph below is a perfect map for a set of independencies $\mathcal{U}$.*

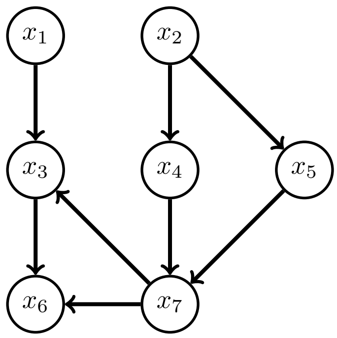


Graph 0

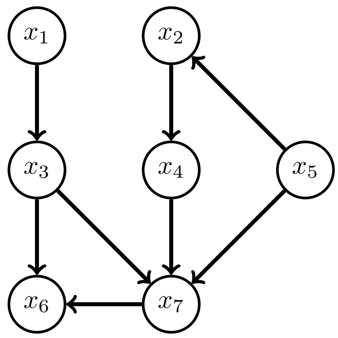
For each of the three graphs below, explain whether the graph is a perfect map, an I-map, or not an I-map for $\mathcal{U}$.



Graph 1



Graph 2



Graph 3

Solution.

- Graph 1 has an immorality $x_2 \to x_5 \leftarrow x_7$ which graph 0 does not have. The graph is thus not I-equivalent to graph 0 and can thus not be a perfect map. Moreover, graph 1 asserts that $x_2 \perp\!\!\!\perp x_7 | x_4$ which is not case for graph 0. Since graph 0 is a perfect map for $\mathcal{U}$, graph 1 asserts an independency that does not hold for $\mathcal{U}$ and can thus not be an I-map for $\mathcal{U}$.
- Graph 2 has an immorality $x_1 \to x_3 \leftarrow x_7$ which graph 0 does not have. Graph 2 thus asserts that $x_1 \perp\!\!\!\perp x_7$, which is not the case for graph 0. Hence, for the same reason as for graph 1, graph 2 is not an I-map for $\mathcal{U}$.

- Graph 3 has the same skeleton and set of immoralities as graph 0. It is thus I-equivalent to graph 0, and hence also a perfect map.