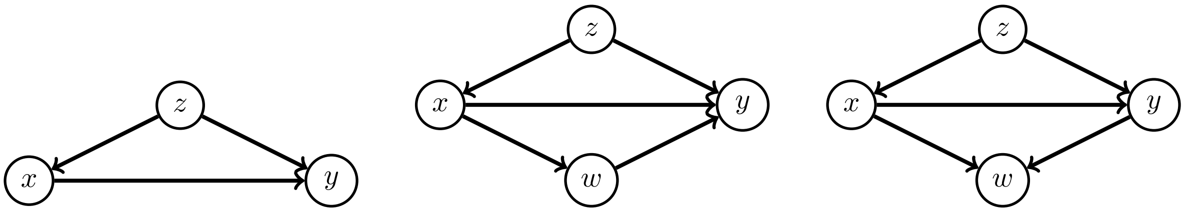
Exercise 1. ***Adjustment for direct causes***

We would like to compute $p(y; do(x) = a)$ for the models represented by the following three causal DAGs. Assume that all variables are discrete.



(a) (b) (c)

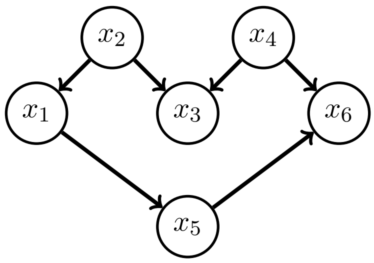
(a) For the causal DAG in (a), specify the factorization of $p(x, y, z; do(x) = a)$ and derive $p(y; do(x) = a)$ in terms of the conditional probability distributions $p(x_i|\text{pa}_i)$ of the graphical model defined the DAG.

(b) For the causal DAG in (b), specify the factorization of $p(x, y, z, w; do(x) = a)$ and derive $p(y; do(x) = a)$ in terms of the conditional probability distributions $p(x_i|\text{pa}_i)$ of the graphical model defined by the DAG.

(c) Answer the same questions for the causal DAG in (c).

Exercise 2. ***Intervening and conditioning***

Consider the following graph G:



For each question below, provide an answer and a brief justification.

(a) Do we have $p(x_6; do(x_1)) = p(x_6|x_1)$?

(b) Do we have $p(x_6|x_3; do(x_1)) = p(x_6|x_1, x_3)$?

(c) Do we have $p(x_6|x_5, x_2; do(x_1)) = p(x_6|x_5)$?

Exercise 3. ***Reject option***

[Murphy PML1 (2022) Ex 5.1] Consider a K-class discrete variable Y with labels $\mathcal{Y} = \{1, \ldots, C\}$. The actions are $\mathcal{A} = \mathcal{Y} \cup \{0\}$, where $a = 0$ denotes the reject option, and choosing action $a = i$ for $i \in \mathcal{Y}$ denotes selecting label i. Define the loss function as follows:

$$\ell(y = j, a = i) = \begin{cases} 0 & \text{if } i = j \text{ and } a \in \{1, \ldots, C\} \\ \lambda_r & \text{if } a = 0 \\ \lambda_c & \text{otherwise,} \end{cases} \tag{1}$$

where λ_r is the cost of a reject, λ_c the cost of an error.

Given information $\mathbf{x}$ we obtain the posterior $p(Y|\mathbf{x})$. Show that the minimum posterior risk is obtained for the following decision rule: we decide $Y = j$ if $p(Y = j|\mathbf{x}) \geq p(Y = k|\mathbf{x})$ for all k (i.e. j is the most probable label) *and* if $p(Y = j|\mathbf{x}) \geq 1 - \lambda_r/\lambda_c$, otherwise we decide to reject.

