Quantum Cyber Security
Lecture 19: Properties of Quantum Systems and Cryptography

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This Lecture

- Idistinguishability: theorem and implication
- No-cloning: theorem and implication
- Monogamy of Entanglement: theorem, implications and measures of entanglement
- Teleportation: what it is and its relation to Quantum One-Time Pad

Distinguishing Pure Quantum States

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  Case I: States $|\psi_i\rangle$ are orthogonal, i.e. $\langle\psi_i|\psi_j\rangle = \delta_{ij}$
  We perform a (projective) measurement that consist of the following operators
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  Exercise: Check that this measurement satisfies the completeness relation
  We can see easily that if the state $|\psi_k\rangle$ is prepared, then $p(i) = \langle\psi_k| P_i |\psi_k\rangle = \delta_{ik}$ and therefore Bob finds with probability one the correct index.

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- Consider two non-orthogonal states $\langle\psi_1|\,\psi_2\rangle \neq 0$
- Related to these are two measurement operators (not necessarily projective) $E_1 = M_1^\dagger M_1$ and $E_2 = M_2^\dagger M_2$
- If we can distinguish them perfectly it means that when Alice sends $|\psi_1\rangle$ Bob has $p(i=1) = \langle\psi_1|\, E_1\, |\psi_1\rangle = 1$ and when Alice sends $|\psi_2\rangle$ Bob has $p(i=2) = \langle\psi_2|\, E_2\, |\psi_2\rangle = 1$

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- From $\sum_i E_i = \mathbb{I}$ and $\langle\psi_1|E_1|\psi_1\rangle = 1$ we conclude that $\langle\psi_1|E_2|\psi_1\rangle = 0$ and thus $\sqrt{E_2}|\psi_1\rangle = 0$

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  Since the two states are non-orthogonal we can write $|\psi_2\rangle = \alpha\,|\psi_1\rangle + \beta\,|\phi\rangle$ where $\langle\psi_1\,|\phi\rangle = 0$ is a unit vector

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  Then it follows : $\langle\psi_2| E_2 |\psi_2\rangle = |\beta|^2 \langle\phi| E_2 |\phi\rangle \leq |\beta|^2 < 1$
  which contradicts our assumption $\square$

Significance of Indistringuishability for Crypto

- B92 QKD protocol relies on this impossibility.
- One can also bound the probability of distinguishing, which is related with how far from orthogonal are the states.
- In many other quantum communication protocols this property is essential (e.g. some protocols that achieve: Quantum Digital Signatures, Quantum Coin-Flipping, Blind Quantum Computing, etc)

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  Unknown state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ and
  $\wedge X = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$

$$\wedge X\,|\psi\rangle\,|0\rangle = a\,|00\rangle + b\,|11\rangle$$

  which is different than: $|\psi\rangle\,|\psi\rangle$

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- No-deleting Theroem: The "time-reversed" version proves that it is impossible to **delete** a qubit using unitary gates.

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- Consider an ancilla initialised at $|0\rangle$, and then the inner product between $|\psi_1\rangle \otimes |0\rangle$ and $|\psi_2\rangle \otimes |0\rangle$:

$$(\langle\psi_1| \otimes \langle 0|)(|\psi_2\rangle \otimes |0\rangle) = \langle\psi_1| \psi_2\rangle \langle 0| 0\rangle = a \tag{1}$$

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- Inner products are invariant under any unitary:

$$\begin{aligned}(\langle\psi_1| \otimes \langle 0|)(|\psi_2\rangle \otimes |0\rangle) &= (\langle\psi_1| \otimes \langle 0|)U^\dagger U(|\psi_2\rangle \otimes |0\rangle) \\ &= (\langle\psi_1| \otimes \langle\psi_1|)(|\psi_2\rangle \otimes |\psi_2\rangle) \\ &= \langle\psi_1|\psi_2\rangle\langle\psi_1|\psi_2\rangle = a^2 \qquad (2)\end{aligned}$$

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- From Eq. (1) and Eq. (2) we have $a = a^2$ possible only if $\langle\psi_1|\,\psi_2\rangle = 1$ or 0 reaching contradiction $\square$

Implications of No-Cloning

- Security of QKD relies on this. If one could copy the BB84 states, then the adversary could measure one copy in each basis, and then compromise the security completely.

- No-Cloning is essential for the indistringuishability too

  Q: Can you come up with a way to distinguish states if you had a copying machine?

- Can put a bound on how well one can copy an unknown quantum state – this is used in certain security proofs

The "maximally" entangled states have some unique properties

1. Perfect correlation: Alice's and Bob's results are perfectly correlated in all bases

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$$

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This is not the case for "partially" entangled e.g.

$$|\psi\rangle = \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle = \frac{1}{2}\left(\sqrt{\frac{2}{3}}(|+\rangle + |-\rangle)(|+\rangle + |-\rangle) + \sqrt{\frac{1}{3}}(|+\rangle - |-\rangle)(|+\rangle - |-\rangle)\right)$$

which clearly is not perfectly correlated

Monogamy of Entanglement

2. Monogamy: If two qubits are maximally entangled, then they are separable with respect to any third qubit

$$\rho_{AB} = \mathrm{Tr}_E(\rho_{ABE}) = |\Phi^+\rangle_{AB}\langle\Phi^+| \;\Rightarrow\; \rho_{ABE} = |\Phi^+\rangle_{AB}\langle\Phi^+| \otimes \rho_E$$

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- Need a measure to quantify how entangled are two subsystems (see later)
- This can be used both to define properly what "**perfect correlation**" means, and to demonstrate that they are **not correlated with third systems**

Implications of Monogamy of Entanglement

- Is the basis for entanglement-based QKD protocols (e.g. BBM92 and E91) security.
- Even for other QKD protocols, their formal security is proven by reduction to entanglement-based protocols.
- Can quantify this since the more quantumly-correlated with one system, the closer it is to being uncorrelated with other systems.

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- A general (for mixed states too) measure of entanglement:
  Relative Entropy of Entanglement: Measures the minimum relative entropy between our state ρ_{AB} and *any* separable state
  $D_{REE}(\rho_{AB}) = \min_{\sigma_{AB} \in \text{ separable states}} S(\rho_{AB} \| \sigma_{AB})$

- **Setting:** Alice and Bob share a pair of entangled qubits

$$|\Phi^+\rangle = \frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}}$$

  There is **no quantum channel** between them (i.e. no quantum state can be physically sent)

  They **can classically communicate**

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- **Task:** Alice wants to send the state $|\psi\rangle$ to Bob

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- The state (before the Bell measurement) can be written as:
  $|\Phi^+\rangle_{AB}|\psi\rangle_C =$
  $1/2[|\Phi^+\rangle_{AC}(a|0\rangle_B + b|1\rangle_B) + |\Phi^-\rangle_{AC}(a|0\rangle_B - b|1\rangle_B) +$
  $+|\Psi^+\rangle_{AC}(a|1\rangle_B + b|0\rangle_B) + |\Psi^-\rangle_{AC}(-a|1\rangle_B + b|0\rangle_B)]$

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- Depending on the outcome, Alice sends to Bob using a classical channel a *"correction"* to make:

$|\Phi^+\rangle_{AC} \rightarrow \mathbb{I}_B$; $|\Phi^-\rangle_{AC} \rightarrow Z_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$|\Psi^+\rangle_{AC} \rightarrow X_B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $|\Psi^-\rangle_{AC} \rightarrow -(ZX)_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

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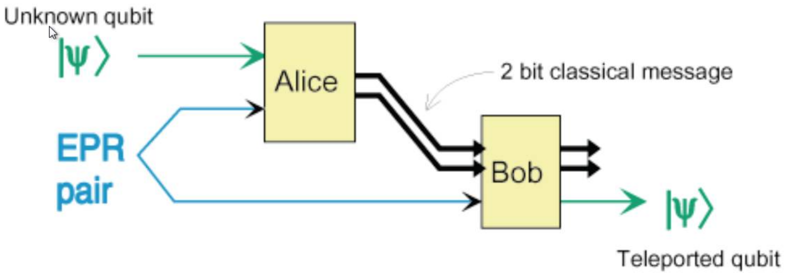
  Note: To complete the teleportation, some corrections are needed which Alice communicates classically to Bob. Otherwise she could "signal" faster than the speed of light!

Pictorially:



Classical Channel + Entanglement = Quantum Channel

Teleportation and QOTP

- Let us label the outcomes as 2-bit string *ab*

$$|\Phi^+\rangle \rightarrow 00 \quad ; \quad |\Phi^-\rangle \rightarrow 01$$
$$|\Psi^+\rangle \rightarrow 10 \quad ; \quad |\Psi^-\rangle \rightarrow 11$$

- We can then rewrite the output state as:

$$X^a Z^b |\psi\rangle$$

- This is really the QOTP where the padding is the outcomes Alice got in her Bell measurement
- The state for Bob (without knowing Alice's outcomes/secret key) is totally random

  Contains no information and thus doesn't violate non-signalling
- Bob cannot know whether Alice has made the measurement (and thus teleportation) or that he holds one side of a Bell pair
- Conversely in QOTP Bob could have received one side of a Bell pair, and not the padded state, thus he has no information!