THE UNIVERSITY of EDINBURGH
informatics

Quantum Cyber Security

Lecture 10: Other QKD and similar protocols

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In this lecture…

We learn about:

- Coin flipping task and motivation
- Classical coin flipping
- Quantum coin flipping
- Simple quantum protocols for coin flipping
- Quantum coin flipping with qutrits
- Bounds on strong coin flipping
- Weak quantum coin flipping
- Implementations of quantum coin flipping

Coin flipping task and motivation

Who gets to pick the music ?

They need a protocol to agree on a random bit
But they don’t trust each other!

This is the task of a coin flipping protocol!
Coin flipping was introduced by Blum in 1983

Definition of (strong) Coin flipping

Definition (Strong) coin flipping**:**

The task of coin flipping consists of two mutually distrustful players, Alice and Bob, and the goal is for both players to output the same random bit $c \in \{0, 1\}$ such that the following properties hold:

1. **Correctness**: if both Alice and Bob are honest then b is uniformly distributed: $p(c = 0) = p(c = 1) = 1/2$.
2. **ε-secure:** neither player can force $p(c = 0) \geq 1/2 + \varepsilon$ or $p(c = 1) \geq 1/2 + \varepsilon$, where $p(c)$ is the probability that the honest player outputs a value c.

The smallest ε for which a protocol is ε-secure is called the **bias.**

Note: Coin flipping is a completely randomized primitive. There is no fixed function that determines the outputs of the players

Question: Is unconditionally secure coin flipping possible?

Unconditionally secure coin flipping?

Impossibility of classical unconditionally secure coin flipping [Blum83]:
No classical coin flipping protocol is secure, i.e. no value of $\varepsilon < 1/2$ can be achieved for security!

If Alice cannot completely bias the output of the protocol, Bob can (and vice versa)

A classical coin flipping protocol

Let's see an example…

1. Alice flips a random bit a ∈ {0, 1} and sends it to Bob

a

b

2. Bob flips a random bit b ∈ {0, 1} and sends it to Alice

3. Both return $c = a \oplus b$

The protocol is correct

But not at all secure!

Question:

Why this protocol is not secure?

Bob sees Alice's message before sending his message, so he can always force his favorite bit value.

Some intuition about the impossibility

There is some sort of asymmetry between Alice and Bob in *any* coin flipping protocol…

The order matters!

The outcome c (regardless of the value) has to be determined **after certain number of rounds** in the protocol consisting of sending some messages. One can always find a message such that, before the message is sent, the outcome is not yet determined, but once the message has been sent it is.

That is why, the player who sends that message (or in other words, **knows more**) has the ability to **bias** the outcome to any possibility.

Classical coin flipping protocols (with computational assumptions)

There exists a coin flipping protocol, assuming perfectly secure One Way Function (OWF) exists [Blum83]

Blum's coin flipping protocol based on OWF:

Alice and Bob agree on some perfectly secure 1-1 OWF **f**, before the protocol

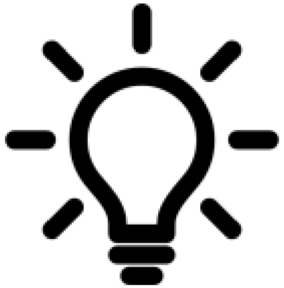
- Alice selects x, sends f(x) to Bob
- Bob needs to guess some non-trivial property of x from f(x) (for instance if x is even or odd) – which he can't better than random because of the properties of OWF, so Bob has to also flip a coin, which will be a random bit
- Then Alice will convince Bob by revealing x

Cleve [Cleve 89,93] showed that for any two-party r-round coin-flipping protocol there exists an efficient adversary that can bias the output of the honest party by $\Omega(1/r)$

Quantum coin flipping

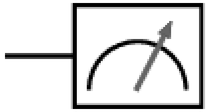
How about quantum coin flipping?
Now let Alice and Bob be **quantum** players!

Question: Why going to quantum regime might help?



The notion of **transcript** in the quantum case is less clear, because quantum states have interesting properties. One cannot easily determine a quantum message

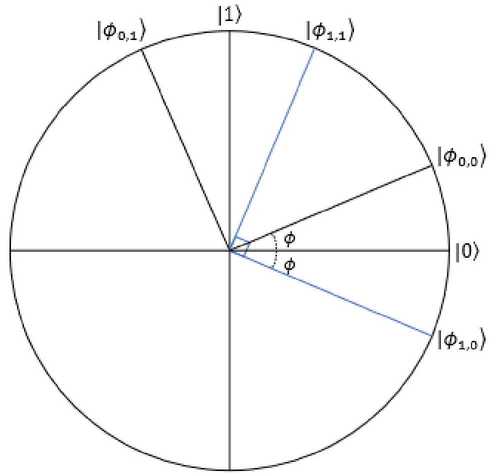
To determine the outcome of a quantum message you need measurements!



Aharonov’s quantum coin flipping protocols

Introduced by Aharonov’s (2000)

$$|\phi_{x,a}\rangle = \begin{cases} |\phi_{x,0}\rangle = \cos\phi|0\rangle + (-1)^{x}\sin\phi\,|1\rangle \\ |\phi_{x,1}\rangle = \sin\phi|0\rangle + (-1)^{x\oplus 1}\cos\phi\,|1\rangle \end{cases}$$



1. Alice selects two random bits x and **a** (a is Alice’s main bit)

2. Alice prepares a state $|\phi_{x,a}\rangle$ based on her choice and sends to Bob

$|\phi_{x,a}\rangle$ →

b ←

3. Bob also selects his random bit **b** and sends to Alice

4.a. Alice sends the random bits to Bob and Bob measures the qubit in the suitable basis

$$c = a \bigoplus b$$

4.b. Bob sends back the qubit and Alice measures and verifies

Security of Aharonov’s coin flipping protocol

Cheating strategies for Alice:
Preparing the wrong states (not according to uniform distribution) to bias the bit, or giving wrong information about (x,a) to Bob

Cheating strategies for Bob:
Try to determine x, a (learn Alice’s bit) before she reveals them classically

Theorem (Aharonov’s protocol security):
The protocol is ε-secure with ε bias at most 0.42

$\Pr[Alice\ win] \leq 0.914$

For optimal $\phi = \frac{\pi}{8}$

$\Pr[Bob\ win] \leq 0.86$

Why it is secure? Let's say Bob is cheating…

Compared to the classical protocol, Alice does not **fully** reveal her bit, before Bob sends his bit.

To bound Bob's success probability, we need to look at the mixture of the state Alice prepares.

$$\rho_{a=0} = \frac{1}{2}(|\phi_{0,0}\rangle\langle\phi_{0,0}| + |\phi_{1,0}\rangle\langle\phi_{1,0}|)$$

$$\rho_{a=1} = \frac{1}{2}(|\phi_{0,1}\rangle\langle\phi_{0,1}| + |\phi_{1,1}\rangle\langle\phi_{1,1}|)$$

Bob needs to know bit $\underline{\underline{a}}$ to cheat.

$\Downarrow$

Bob needs to distinguish between these two cases.

This is a "State discrimination problem" between two density matrices (ensembles).

To do this, one needs to find the best POVM, that has the least error to distinguish between the two states.

Holevo-Helstrom bound: The optimal probability of distinguishing between two density matrices which have been picked with equal probability, is given by this bound:

$$P_{disc}^{opt} = \frac{1}{2} + \frac{1}{4}\|\rho_1 - \rho_2\|_{tr}$$

Let's calculate the Holevo bound!

$$\begin{cases} |\phi_{00}\rangle = \cos\varphi|0\rangle + \sin\varphi|1\rangle \\ |\phi_{10}\rangle = \cos\varphi|0\rangle - \sin\varphi|1\rangle \\ |\phi_{01}\rangle = \sin\varphi|0\rangle - \cos\varphi|1\rangle \\ |\phi_{11}\rangle = \sin\varphi|0\rangle + \cos\varphi|1\rangle \end{cases}$$

$$\rho_0 = \frac{1}{2}\left(|\phi_{00}\rangle\langle\phi_{00}| + |\phi_{10}\rangle\langle\phi_{10}|\right)$$

$$= \frac{1}{2}\Big(\cos^2\varphi\,|0\rangle\langle 0| + \sin^2\varphi\,|1\rangle\langle 1| + \sin\varphi\cos\varphi\,(|0\rangle\langle 1| + |1\rangle\langle 0|) +$$

$$\cos^2\varphi\,|0\rangle\langle 0| + \sin^2\varphi\,|1\rangle\langle 1| - \sin\varphi\cos\varphi\,(|0\rangle\langle 1| + |1\rangle\langle 0|)\Big)$$

$$= \cos^2\varphi\,|0\rangle\langle 0| + \sin^2\varphi\,|1\rangle\langle 1|$$

Similarly $\rightarrow$ $\rho_1 = \sin^2\varphi\,|0\rangle\langle 0| + \cos^2\varphi\,|1\rangle\langle 1|$

$$\rho_0 - \rho_1 = (\cos^2\varphi - \sin^2\varphi)|0\rangle\langle 0| + (\sin^2\varphi - \cos^2\varphi)|1\rangle\langle 1| = \begin{pmatrix} \cos 2\varphi & 0 \\ 0 & -\cos 2\varphi \end{pmatrix} \rightarrow \begin{array}{l} \lambda_1 = \cos 2\varphi \\ \lambda_2 = -\cos 2\varphi \end{array}$$

$$\|\rho_0 - \rho_1\|_{tr} = \sum_i |\lambda_i| = 2\cos 2\varphi$$

$$\|\rho_{a=0} - \rho_{a=1}\| = 2\cos 2\varphi \xrightarrow[\text{optimal } \varphi = \pi/8]{} Pr[\text{Bob cheat}] \leq \frac{1}{2} + \frac{\cos 2\phi}{2} \approx \underline{\underline{0.853}}$$

Quantum coin flipping with qutrits

Another similar protocol was introduced by Ambainis (2004) with a better bias:

$$|\phi_{a,x}\rangle = \begin{cases} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & a = 0, x = 0 \\ \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & a = 0, x = 1 \\ \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) & a = 1, x = 0 \\ \frac{1}{\sqrt{2}}(|0\rangle - |2\rangle) & a = 1, x = 1 \end{cases}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Ambainis's coin flipping protocol:

1. Alice selects $x \in \{0, 1\}$ and $a \in \{0, 1\}$ uniformly at random and sends $|\phi_{a,x}\rangle$ to Bob.
2. Bob selects $b \in \{0, 1\}$ uniformly at random and sends b to Alice.
3. Alice sends a and x to Bob.
4. Bob verifies the state he received from Alice in step 1. is $|\phi_{a,x}\rangle$, if it is not the case then he declares that Alice has been cheating and aborts the protocol.
5. Both players return the outcome $c = a \oplus b$

Security of Ambainis coin flipping protocol

Theorem (Security of Ambainis protocol):
The protocol is ε-secure with ε bias 0.25

As an exercise, compute the mixed state ρ_0 and ρ_1 for this protocol similar to the previous case (They are 3x3 density matrices).

Holevo bound $\rightarrow$ $Pr[\text{Bob cheating}] = \frac{1}{2} + \frac{1}{4} \times 1 = \frac{3}{4}$

How about Alice cheating?

Bounding success probability of Alice is harder because she can prepare arbitrary states (for instance entangled with other information), which she can use later to cheat, after Bob reveals.

To bound Alice's success probability, we need to "symmetrize" Alice's strategy. There is a strategy for dishonest Alice, leading to a density matrix of certain form for which Alice achieves a=b with the same probability.

But again, it can be shown that for that symmetric density matrix Alice can at most cheat with probability ¾

Perfectly secure quantum coin flipping?

Can we do much better? Can we design a quantum protocol that achieves arbitrary small bias using quantum information?

No!
perfectly secure strong coin-flipping is also impossible for quantum protocols.

Kitaev's bound for strong coin flipping:
The smallest bias any strong coin flipping protocol can achieve is

$$\varepsilon = \frac{\sqrt{2} - 1}{2} \approx 0.207.$$

The proof is not easy… It relies on Linear Programing (LP) and Semidefinite Programming (SDP).

Weak quantum coin flipping

If the choice of Alice and Bob is pre-determined, we can relax the security requirement:

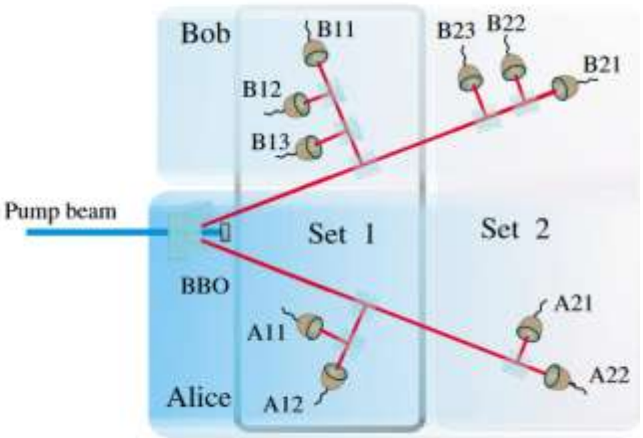
Weak coin flipping:
Weak coin flipping is similar to strong coin flipping except that we only require that malicious **Alice** cannot force p(**c = 0**) ≥ 1/2 + ε, and malicious **Bob** cannot force p(**c = 1**) ≥ 1/2 + ε

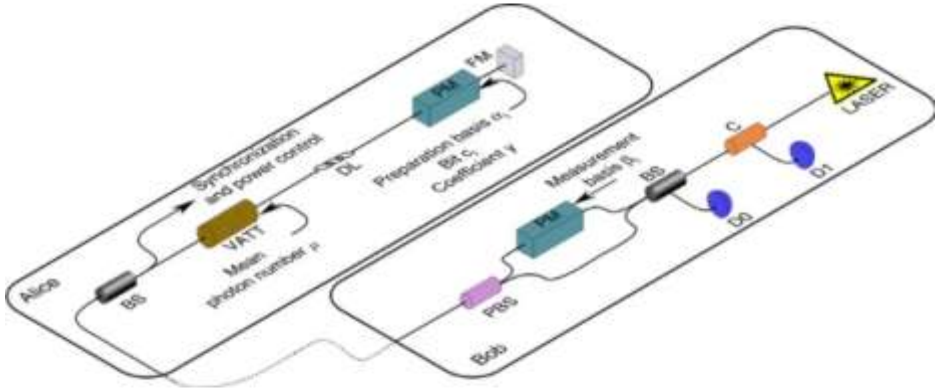
It has been shown that weak quantum coin flipping, with arbitrarily small (but non-zero) bias ε, is possible (by Mochon in 2007)

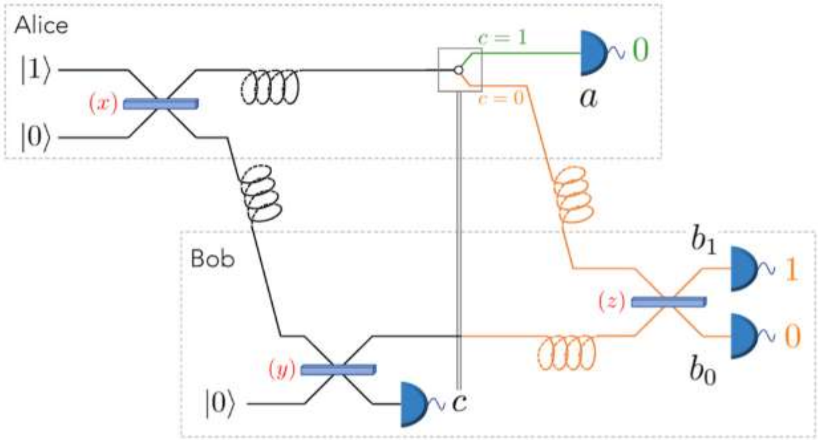
But the protocol that achieves this arbitrary small bias is complicated and requires multiple rounds that scales exponentially with $\frac{1}{\varepsilon}$
Designing concrete protocols with arbitrary small bias is still open research problem (Almost solved by Arora et al. (2019) for protocols with bias around 1/10)

Can we implement these protocols?

Just to satisfy your curiosity ;)







Implementation of Ambainis' protocol:
Molina-Terriza, G., Vaziri, A., Ursin, R., & Zeilinger, A. (2005). Experimental quantum coin tossing. *PRL*

Implementation of a practical coin flipping protocol by Pappa & Chailloux:
Pappa, A., Jouguet, P., Lawson, T., Chailloux, A., Legré, M., Trinkler & Diamanti, E. (2014). Experimental plug and play quantum coin flipping. *Nature communications*

Implementation of weak coin flipping:
Bozzio, M., Chabaud, U., Kerenidis, I., & Diamanti, E. (2020). Quantum weak coin flipping with a single photon. *PRA*

Resources

1. Introduction to Quantum Cryptography by *Thomas Vidick and Stephanie Wehner*: chapter 10, 10.1

Extra materials:

- [Blu83] Manuel Blum. "Coin flipping by telephone a protocol for solving impossible prob- lems". In: ACM SIGACT News 15.1 (1983), pages 23–27.
- [Cle+86] R. Cleve. Limits on the security of coin flips when half the processors are faulty. In Proceedings of the 18th Annual ACM Symposium on Theory of Computing, pages 364–369, 1986.
- [Cle+93] Cleve R, Impagliazzo R. Martingales, collective coin flipping and discrete control processes. other words. 1993 Nov;1(5):8.
- [Aha+00] Dorit Aharonov et al. "Quantum bit escrow". In: Proceedings of the thirty-second annual ACM symposium on Theory of computing. ACM. 2000, pages 705–714
- [Amb01] Andris Ambainis. "A new protocol and lower bounds for quantum coin flipping". In: Proceedings of the thirty-third annual ACM symposium on Theory of computing. ACM. 2001, pages 134–142.
- [GW07] Gus Gutoski and John Watrous. "Toward a general theory of quantum games". In: Proceedings of the thirty-ninth annual ACM symposium on Theory of computing. ACM. 2007, pages 565–574
- [Moc07] Carlos Mochon. "Quantum weak coin flipping with arbitrarily small bias". In: arXiv preprint arXiv:0711.4114 (2007)
- [Aha+16] Dorit Aharonov et al. "A Simpler Proof of the Existence of Quantum Weak Coin Flipping with Arbitrarily Small Bias". In: SIAM Journal on Computing 45.3 (2016), pages 633–679.
- [CK09] André Chailloux and Iordanis Kerenidis. "Optimal quantum strong coin flipping". In: Foundations of Computer Science, 2009. FOCS'09. 50th Annual IEEE Symposium on. IEEE. 2009, pages 527–533.
- [Aro+19] Arora AS, Roland J, Weis S. Quantum weak coin flipping. InProceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing 2019 (pp. 205-216).