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Tutorial 5: Solutions

Problem 1

Consider the four Bell states

$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad \left|\Phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$\left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad \left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Those maximally entangled states form an orthonormal basis of the two-qubit Hilbert space $\mathcal{H} = \mathbb{C}^4$.

(a) Verify that the Bell states form an orthonormal family of states, i.e., that they are pair-wise orthogonal, and each of them is normalised.

Solution. We need to prove that for any two distinct Bell states $|\Psi\rangle, |\Psi'\rangle$, they are orthogonal, i.e., $\langle\Psi|\Psi'\rangle = 0$. As an example we consider $|\Phi\rangle$ and $|\Phi'\rangle$ but the derivation is similar for other pairs:

$$\begin{aligned}\left\langle\Phi^+\middle|\Phi^-\right\rangle &= \frac{\langle 00| + \langle 11|}{\sqrt{2}} \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{\langle 00|00\rangle - \langle 00|11\rangle + \langle 11|00\rangle - \langle 11|11\rangle}{2} \\ &= \frac{1 - 0 + 0 - 1}{2} = 0\end{aligned}$$

where we have used the fact that $\{|0\rangle, |1\rangle\}$ is an orthonormal basis. Let us show that $|\Phi^+\rangle$ has norm 1:

$$\begin{aligned}\left\langle\Phi^+\middle|\Phi^+\right\rangle &= \frac{\langle 00| + \langle 11|}{\sqrt{2}} \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{\langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle + \langle 11|11\rangle}{2} \\ &= \frac{1 + 0 + 0 + 1}{2} = 1\end{aligned}$$

A similar derivation for other Bell states then allows us to conclude that all of them are normalised. Since they are also pair-wise orthogonal, they form an orthonormal family of states.

(b) Simplify the following:

i. $X \otimes X\,|\Psi^-\rangle$

Solution.

$$X \otimes X \left|\Psi^-\right\rangle = X \otimes X \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{X \otimes X\,|01\rangle - X \otimes X\,|10\rangle}{\sqrt{2}} = \frac{|10\rangle - |01\rangle}{\sqrt{2}} = -\left|\Psi^-\right\rangle$$

ii. $X \otimes Z\,|\Psi^-\rangle$

Solution.

$$X \otimes Z \left|\Psi^-\right\rangle = \frac{X \otimes Z\,|01\rangle - X \otimes Z\,|10\rangle}{\sqrt{2}} = \frac{-|11\rangle - |00\rangle}{\sqrt{2}} = -\left|\Phi^+\right\rangle$$

iii. $Z \otimes X\,|\Psi^-\rangle$

Solution.

$$Z \otimes X \left|\Psi^-\right\rangle = \frac{Z \otimes X\,|01\rangle - Z \otimes X\,|10\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \left|\Phi^+\right\rangle$$

iv. $Z \otimes Z |\Psi^-\rangle$

Solution.

$$Z \otimes Z \left|\Psi^-\right\rangle = \frac{Z \otimes Z |01\rangle - Z \otimes Z |10\rangle}{\sqrt{2}} = \frac{-|01\rangle + |10\rangle}{\sqrt{2}} = -\left|\Psi^-\right\rangle$$

Problem 2

Consider the CHSH "game" described in the lecture. Assume that Alice and Bob share the quantum state

$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Recall that

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $x = 0$ Alice measures the observable $A_0 = Z$, and if $x = 1$ Alice measures the observable $A_1 = X$. If $y = 0$ Bob measures the observable $B_0 = \frac{1}{\sqrt{2}}(X + Z)$, and if $y = 1$ Bob measures the observable $B_1 = \frac{1}{\sqrt{2}}(X - Z)$.

(a) Compute the correlator $E_{00} = \langle\Phi^+| A_0 \otimes B_0 |\Phi^+\rangle = \langle\Phi^+| Z \otimes \frac{1}{\sqrt{2}}(X + Z) |\Phi^+\rangle$.

Solution.

$$\begin{aligned} E_{00} &= \left\langle\Phi^+\right| A_0 \otimes B_0 \left|\Phi^+\right\rangle = \left\langle\Phi^+\right| Z \otimes \frac{1}{\sqrt{2}} (X + Z) \left|\Phi^+\right\rangle \\ &= \frac{1}{\sqrt{2}} \left\langle\Phi^+\right| Z \otimes X \left|\Phi^+\right\rangle + \frac{1}{\sqrt{2}} \left\langle\Phi^+\right| Z \otimes Z \left|\Phi^+\right\rangle. \end{aligned}$$

Let us define $D = \langle\Phi^+| Z \otimes X |\Phi^+\rangle$ and $E = \langle\Phi^+| Z \otimes Z |\Phi^+\rangle$ so that $E_{00} = \frac{1}{\sqrt{2}}(D + E)$. We first compute the term D. To compute D, we start from

$$\begin{aligned} Z \otimes X \left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}(Z \otimes X)(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}[(Z \otimes X) |00\rangle + (Z \otimes X) |11\rangle] \\ &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned}$$

Then, we have

$$\begin{aligned} D = \left\langle\Phi^+\right| Z \otimes X \left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}} \left\langle\Phi^+\right| (|01\rangle - |10\rangle) \\ &= \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle - |10\rangle) = 0. \end{aligned}$$

To compute E, we start from

$$\begin{aligned} Z \otimes Z \left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}(Z \otimes Z)(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}[(Z \otimes Z) |00\rangle + (Z \otimes Z) |11\rangle] \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \end{aligned}$$

Then, we have

$$E = \left\langle \Phi^+ \right| Z \otimes Z \left| \Phi^+ \right\rangle = \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| (|00\rangle + |11\rangle)$$
$$= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle) = 1.$$

Therefore, we obtain

$$E_{00} = \frac{D+E}{\sqrt{2}} = \frac{0+1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

(b) Compute the quantity $\beta = E_{00} - E_{01} + E_{10} + E_{11}$ and show that it attains the maximum Bell inequality violation $2\sqrt{2}$, as given in the lectures.

Solution. We will proceed similarly to the previous part to compute E_{01}, E_{10}, and E_{11}. First, we see that

$$E_{01} = \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| Z \otimes X \left| \Phi^+ \right\rangle - \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| Z \otimes Z \left| \Phi^+ \right\rangle$$
$$= \frac{D-E}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

In the case of E_{10}, we have

$$E_{10} = \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| X \otimes X \left| \Phi^+ \right\rangle + \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| X \otimes Z \left| \Phi^+ \right\rangle.$$

Let us define $F = \langle \Phi^+ | X \otimes X | \Phi^+ \rangle$ and $G = \langle \Phi^+ | X \otimes Z | \Phi^+ \rangle$ so that $E_{10} = \frac{1}{\sqrt{2}}(F + G)$. We first compute the term F. To compute F, we start from

$$X \otimes X \left| \Phi^+ \right\rangle = \frac{1}{\sqrt{2}}(X \otimes X)(|00\rangle + |11\rangle)$$
$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Then, we have

$$F = \left\langle \Phi^+ \right| X \otimes X \left| \Phi^+ \right\rangle = \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| (|00\rangle + |11\rangle)$$
$$= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle) = 1.$$

To compute G, we start from

$$X \otimes Z \left| \Phi^+ \right\rangle = \frac{1}{\sqrt{2}}(X \otimes Z)(|00\rangle + |11\rangle)$$
$$= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle).$$

Then, we have

$$G = \left\langle \Phi^+ \right| X \otimes Z \left| \Phi^+ \right\rangle = \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| (|10\rangle - |01\rangle)$$
$$= \frac{1}{2}(\langle 00| + \langle 11|)(|10\rangle - |01\rangle) = 0$$

Therefore, we obtain

$$E_{10} = \frac{F+G}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

For the case of E_{11}, we have

$$\begin{aligned} E_{11} &= \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| X \otimes X \left| \Phi^+ \right\rangle - \frac{1}{\sqrt{2}} \left\langle \Phi^+ \right| X \otimes Z \left| \Phi^+ \right\rangle \\ &= \frac{F-G}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Combining all of these, we obtain the quantity

$$\beta = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\sqrt{2}.$$

Problem 3

Consider the same setting of the game as in Problem 1, but with the difference that now Alice and Bob share the state $|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$. Compute the quantity β in this case.

Solution. We start with the expressions for each correlator

$$\begin{aligned} E_{00} &= \langle\psi| Z \otimes \frac{1}{\sqrt{2}}(X+Z)|\psi\rangle, \\ E_{01} &= \langle\psi| Z \otimes \frac{1}{\sqrt{2}}(X-Z)|\psi\rangle, \\ E_{10} &= \langle\psi| X \otimes \frac{1}{\sqrt{2}}(X+Z)|\psi\rangle, \\ E_{11} &= \langle\psi| X \otimes \frac{1}{\sqrt{2}}(X-Z)|\psi\rangle. \end{aligned}$$

For E_{00}, we have

$$E_{00} = \frac{1}{\sqrt{2}}(\langle\psi| Z \otimes X |\psi\rangle + \langle\psi| Z \otimes Z |\psi\rangle)$$

To compute the first term $D = \langle\psi| Z \otimes X |\psi\rangle$, we first use the fact that

$$Z \otimes X |\psi\rangle = (Z \otimes X)\frac{1}{\sqrt{3}}|00\rangle + (Z \otimes X)\frac{\sqrt{2}}{\sqrt{3}}|11\rangle = \frac{1}{\sqrt{3}}|01\rangle - \frac{\sqrt{2}}{\sqrt{3}}|10\rangle,$$

which gives us

$$D = \left(\frac{1}{\sqrt{3}}\langle 00| + \frac{\sqrt{2}}{\sqrt{3}}\langle 11|\right)\left(\frac{1}{\sqrt{3}}|01\rangle - \frac{\sqrt{2}}{\sqrt{3}}|10\rangle\right) = 0.$$

Dor the second term $E = \langle\psi| Z \otimes Z |\psi\rangle$,

$$Z \otimes Z |\psi\rangle = (Z \otimes Z)\frac{1}{\sqrt{3}}|00\rangle + (Z \otimes Z)\frac{\sqrt{2}}{\sqrt{3}}|11\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{\sqrt{2}}{\sqrt{3}}|11\rangle = |\psi\rangle,$$

which gives us

$$E = \langle\psi|\psi\rangle = 1.$$

Thus, we have

$$E_{00} = \frac{D+E}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

From the computed quantities, we can also deduce directly that

$$E_{01} = \frac{1}{\sqrt{2}} \langle\psi| Z \otimes X |\psi\rangle - \frac{1}{\sqrt{2}} \langle\psi| Z \otimes Z |\psi\rangle = \frac{D-E}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

For E_{10}, we have

$$E_{10} = \frac{1}{\sqrt{2}} \langle\psi| X \otimes X |\psi\rangle + \frac{1}{\sqrt{2}} \langle\psi| X \otimes Z |\psi\rangle .$$

To compute $F = \langle\psi| X \otimes X |\psi\rangle$, we start from

$$X \otimes X |\psi\rangle = \frac{1}{\sqrt{3}}(X \otimes X)|00\rangle + \frac{\sqrt{2}}{\sqrt{3}}(X \otimes X)|11\rangle = \frac{1}{\sqrt{3}}|11\rangle + \frac{\sqrt{2}}{\sqrt{3}}|00\rangle ,$$

which gives us

$$F = \left(\frac{1}{\sqrt{3}}\langle 00| + \frac{\sqrt{2}}{\sqrt{3}}\langle 11|\right)\left(\frac{1}{\sqrt{3}}|11\rangle + \frac{\sqrt{2}}{\sqrt{3}}|00\rangle\right) = \frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}.$$

To compute $G = \langle\psi| X \otimes Z |\psi\rangle$, we start from

$$X \otimes Z |\psi\rangle = \frac{1}{\sqrt{3}}(X \otimes Z)|00\rangle + \frac{\sqrt{2}}{\sqrt{3}}(X \otimes Z)|11\rangle = \frac{1}{\sqrt{3}}|10\rangle - \frac{\sqrt{2}}{\sqrt{3}}|01\rangle ,$$

which gives us

$$G = \left(\frac{1}{\sqrt{3}}\langle 00| + \frac{\sqrt{2}}{\sqrt{3}}\langle 11|\right)\left(\frac{1}{\sqrt{3}}|10\rangle - \frac{\sqrt{2}}{\sqrt{3}}|01\rangle\right) = 0.$$

Thus, we have

$$E_{10} = \frac{F+G}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{3} = \frac{2}{3}.$$

From the computed quantities, we can also deduce directly that

$$E_{11} = \frac{F-G}{\sqrt{2}} = \frac{2}{3}.$$

Finally, we obtain the quantity

$$\beta = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) + \frac{2}{3} + \frac{2}{3} = \sqrt{2} + \frac{4}{3} \approx 2.74 < 2\sqrt{2}.$$

Problem 4

Consider the same setting of the game as in Problem 1, but now Alice and Bob share a mixed state ρ that is given by the ensemble where with probability $p_1 = 1/4$ the state is

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

with probability $p_2 = 1/4$ the state is

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle,$$

and with probability $p_3 = 1/2$ the state is

$$|\psi_3\rangle = |0\rangle \otimes |+\rangle.$$

(a) Compute the correlator $E_{01}(\rho) = \mathrm{tr}[\rho(Z \otimes \frac{X-Z}{\sqrt{2}})]$.

Solution. First notice that

$$\begin{aligned}
E_{01}(\rho) &= \mathrm{tr}(\rho A_0 \otimes B_1) \\
&= \mathrm{tr}[(p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2| + p_3 |\psi_3\rangle\langle\psi_3|)A_0 \otimes B_0] \\
&= p_1 \,\mathrm{tr}(|\psi_1\rangle\langle\psi_1| A_0 \otimes B_0) + p_2 \,\mathrm{tr}(|\psi_2\rangle\langle\psi_2| A_0 \otimes B_0) + p_3 \,\mathrm{tr}(|\psi_3\rangle\langle\psi_3| A_0 \otimes B_0) \\
&= p_1 E_{01}(|\psi_1\rangle\langle\psi_1|) + p_2 E_{01}(|\psi_2\rangle\langle\psi_2|) + p_3 E_{01}(|\psi_3\rangle\langle\psi_3|).
\end{aligned}$$

As part of Problem 1, we calculated $E_{01}(|\psi_1\rangle\langle\psi_1|) = -\frac{1}{\sqrt{2}}$. As part of Problem 2, we calculated $E_{01}(|\psi_2\rangle\langle\psi_2|) = -\frac{1}{\sqrt{2}}$. We now calculate the third term

$$E_{01}(|\psi_3\rangle\langle\psi_3|) = \frac{1}{\sqrt{2}}\langle\psi_3| Z \otimes X |\psi_3\rangle - \frac{1}{\sqrt{2}}\langle\psi_3| Z \otimes Z |\psi_3\rangle.$$

To do this, note that

$$Z \otimes X |\psi_3\rangle = Z \otimes X \left(\frac{|00\rangle + |01\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|00\rangle = |\psi_3\rangle$$

and thus

$$\langle\psi_3| Z \otimes X |\psi_3\rangle = \langle\psi_3|\psi_3\rangle = 1.$$

Also note that

$$Z \otimes Z |\psi_3\rangle = Z \otimes Z \left(\frac{|00\rangle + |01\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|01\rangle$$

and thus

$$\langle\psi_3| Z \otimes Z |\psi_3\rangle = \frac{1}{2}(\langle 00| + \langle 01|)(|00\rangle - |01\rangle) = 0.$$

We therefore have

$$E_{01}(|\psi_3\rangle\langle\psi_3|) = \frac{1-0}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Finally, combining $E_{01}(|\psi_1\rangle\langle\psi_1|)$, $E_{01}(|\psi_2\rangle\langle\psi_2|)$, and $E_{01}(|\psi_3\rangle\langle\psi_3|)$ we obtain

$$E_{01}(\rho) = \frac{1}{4}\cdot\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\cdot\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\cdot\left(\frac{1}{\sqrt{2}}\right) = 0.$$

(b) Determine the quantity β corresponding to this realisation of the CHSH game.

Solution. In the previous part, we showed $E_{01}(\rho) = 0$. Similarly to the previous part,

$$
\begin{aligned}
E_{00}(\rho) &= p_1 E_{00}(|\psi_1\rangle\langle\psi_1|) + p_2 E_{00}(|\psi_2\rangle\langle\psi_2|) + p_3 E_{00}(|\psi_3\rangle\langle\psi_3|),\\
E_{10}(\rho) &= p_1 E_{10}(|\psi_1\rangle\langle\psi_1|) + p_2 E_{10}(|\psi_2\rangle\langle\psi_2|) + p_3 E_{10}(|\psi_3\rangle\langle\psi_3|),\\
E_{11}(\rho) &= p_1 E_{11}(|\psi_1\rangle\langle\psi_1|) + p_2 E_{11}(|\psi_2\rangle\langle\psi_2|) + p_3 E_{11}(|\psi_3\rangle\langle\psi_3|).
\end{aligned}
$$

From Problem 1 we know

$$E_{00}(|\psi_1\rangle\langle\psi_1|) = E_{10}(|\psi_1\rangle\langle\psi_1|) = E_{11}(|\psi_1\rangle\langle\psi_1|) = \frac{1}{\sqrt{2}}$$

and from Problem 2 we know

$$E_{00}(|\psi_2\rangle\langle\psi_2|) = \frac{1}{\sqrt{2}}, \quad E_{10}(|\psi_2\rangle\langle\psi_2|) = E_{11}(|\psi_2\rangle\langle\psi_2|) = \frac{2}{3}.$$

Hence, we only need to compute $E_{00}(|\psi_3\rangle\langle\psi_3|)$, $E_{10}(|\psi_3\rangle\langle\psi_3|)$, and $E_{11}(|\psi_3\rangle\langle\psi_3|)$. For $E_{00}(|\psi_3\rangle\langle\psi_3|)$, we can use quantities calculated in the previous part to write

$$E_{00}(|\psi_3\rangle\langle\psi_3|) = \frac{1}{\sqrt{2}} \langle\psi_3| Z \otimes X |\psi_3\rangle + \frac{1}{\sqrt{2}} \langle\psi_3| Z \otimes Z |\psi_3\rangle = \frac{1+0}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

For $E_{10}(|\psi_3\rangle\langle\psi_3|)$, we compute

$$E_{00}(|\psi_3\rangle\langle\psi_3|) = \frac{1}{\sqrt{2}} \langle\psi_3| X \otimes X |\psi_3\rangle + \frac{1}{\sqrt{2}} \langle\psi_3| X \otimes Z |\psi_3\rangle .$$

Noting that

$$X \otimes X |\psi_3\rangle = X \otimes X \left(\frac{|00\rangle + |01\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} |11\rangle + \frac{1}{\sqrt{2}} |10\rangle$$

so that

$$\langle\psi_3| X \otimes X |\psi_3\rangle = \frac{1}{2}(\langle 00| + \langle 01|)(|11\rangle + |10\rangle) = 0,$$

and

$$X \otimes Z |\psi_3\rangle = X \otimes Z \left(\frac{|00\rangle + |01\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} |10\rangle - \frac{1}{\sqrt{2}} |11\rangle$$

so that

$$\langle\psi_3| X \otimes Z |\psi_3\rangle = \frac{1}{2}(\langle 00| + \langle 01|)(|10\rangle - |11\rangle) = 0.$$

Therefore, we obtain $E_{10}(|\psi_3\rangle\langle\psi_3|) = \frac{0+0}{\sqrt{2}} = 0$. Similarly, for $E_{11}(|\psi_3\rangle\langle\psi_3|)$ we compute

$$E_{11}(|\psi_3\rangle\langle\psi_3|) = \frac{1}{\sqrt{2}} \langle\psi_3| X \otimes X |\psi_3\rangle - \frac{1}{\sqrt{2}} \langle\psi_3| X \otimes Z |\psi_3\rangle = \frac{0-0}{\sqrt{2}} = 0.$$

We now combine all computed quantities to obtain

$$
\begin{aligned}
E_{00}(\rho) &= \frac{1}{4} \cdot \frac{1}{\sqrt{2}} + \frac{1}{4} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}},\\
E_{10}(\rho) &= \frac{1}{4} \cdot \frac{1}{\sqrt{2}} + \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot 0 = \frac{1}{4\sqrt{2}} + \frac{1}{6},\\
E_{11}(\rho) &= \frac{1}{4} \cdot \frac{1}{\sqrt{2}} + \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot 0 = \frac{1}{4\sqrt{2}} + \frac{1}{6}.
\end{aligned}
$$

Finally, we combine the four correlators to obtain

$$\begin{aligned}\beta &= E_{00}(\rho) - E_{01}(\rho) + E_{10}(\rho) + E_{11}(\rho) \\ &= \frac{1}{\sqrt{2}} - 0 + \left(\frac{1}{4\sqrt{2}} + \frac{1}{6}\right) + \left(\frac{1}{4\sqrt{2}} + \frac{1}{6}\right) \\ &= \frac{3}{2\sqrt{2}} + \frac{1}{3} \approx 1.394.\end{aligned}$$

Problem 5

In this problem, we will derive the Schmidt decomposition for a two-qubit bipartite system. That is, for any two-qubit bipartite state $|\psi\rangle_{AB}$, there exist orthonormal bases $\{|e_1\rangle, |e_2\rangle\}$ and $\{|f_1\rangle, |f_2\rangle\}$ for the single-qubit systems A and B respectively and positive constants c_1 and c_2 such that $|\psi\rangle_{AB} = c_1 |e_1\rangle \otimes |f_1\rangle + c_2 |e_2\rangle \otimes |f_2\rangle$.

(a) Consider any two orthonormal bases for the systems A and B, $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ respectively. Write $|\psi\rangle_{AB}$ in the matrix representation M_ψ with respect to these bases.

Solution. We can write the state with respect to the bases $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ as

$$|\psi\rangle_{AB} = m_{11} |a_1\rangle \otimes |b_1\rangle + m_{12} |a_1\rangle \otimes |b_2\rangle + m_{21} |a_2\rangle \otimes |b_1\rangle + m_{22} |a_2\rangle \otimes |b_2\rangle$$

This then gives us the matrix

$$M_\psi = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

(b) Consider the singular value decomposition $M_\psi = U\Sigma V^\dagger$, where U and V are 2×2 unitaries and Σ is a diagonal matrix whose entries are the joint eigenvalues of the bipartite system. From this decomposition deduce $\{|e_1\rangle, |e_2\rangle\}$ and $\{|f_1\rangle, |f_2\rangle\}$. What are the constants c_1 and c_2?

Solution. Using the singular value decomposition for M_ψ, we have

$$M_\psi = U\Sigma V^\dagger.$$

Given that U and V are 2×2 matrices and Σ is a diagonal matrix, we can write them as

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \quad V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Then, we can write M_ψ as

$$M_\psi = \begin{pmatrix} u_{11}d_1v_{11}^* + u_{12}d_2v_{12}^* & u_{11}d_1v_{21}^* + u_{12}d_2v_{22}^* \\ u_{21}d_1v_{11}^* + u_{22}d_2v_{12}^* & u_{21}d_1v_{21}^* + u_{22}d_2v_{22}^* \end{pmatrix},$$

which can alternatively be rewritten as

$$\begin{aligned}M_\psi &= d_1 \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} \begin{pmatrix} v_{11}^* & v_{21}^* \end{pmatrix} + d_2 \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} \begin{pmatrix} v_{12}^* & v_{22}^* \end{pmatrix} \\ &= d_1 \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}^\dagger + d_2 \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}^\dagger\end{aligned}$$

Now, if we use the notation $e_1 = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix}$, $e_2 = \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix}$, $f_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$, and $f_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$, then we can write

$$M_\psi = d_1 \cdot e_1 f_1^\dagger + d_2 \cdot e_2 f_2^\dagger.$$

Therefore, finally we have

$$|\psi\rangle_{AB} = d_1 \cdot e_1 \otimes f_1 + d_2 \cdot e_2 \otimes f_2.$$

Example. We now show here an example of the Schmidt decomposition. Take the two initial bases to be computational bases $\{|e_1\rangle, |e_2\rangle\} = \{|f_1\rangle, |f_2\rangle\} = \{|0\rangle, |1\rangle\}$. Let the state $|\psi\rangle_{AB}$ be described with respect to these bases as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|11\rangle .$$

Then, we have

$$M_\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Now, we need to find the singular value decomposition for this matrix $M_\psi = U\Sigma V^\dagger$. To do this, the steps are the following:

i. Compute $W_1 = MM^\intercal$ and $W_2 = M^\intercal M$.
ii. Determine the eigenvalues of W_1 and W_2 and the eigenvectors corresponding to these eigenvalues.
iii. Normalise the eigenvectors. Then, the normalized eigenvectors corresponding to W_1 are the columns of U and the normalized eigenvectors corresponding to W_2 are the columns of V.
iv. The elements on the diagonal of Σ, placed in descending order, are the square roots of the eigenvalues of W_1 (or W_2).

We have the matrices

$$W_1 = \begin{pmatrix} \frac{5}{6} & \frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} & \frac{1}{6} \end{pmatrix}, \quad W_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}.$$

For W_1, we have the eigenvalues $\lambda_1 = \frac{3+\sqrt{6}}{6}$ and $\lambda_2 = \frac{3-\sqrt{6}}{6}$, and the corresponding eigenvectors

$$u_1 = \begin{pmatrix} \sqrt{2}+\sqrt{3} \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} \sqrt{2}-\sqrt{3} \\ 1 \end{pmatrix}.$$

For W_2, we have the eigenvalues $\lambda_1 = \frac{3+\sqrt{6}}{6}$ and $\lambda_2 = \frac{3-\sqrt{6}}{6}$ and the corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

After normalising the eigenvectors, we obtain U and V as

$$U = \begin{pmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}+\sqrt{3}}{\sqrt{1+(\sqrt{2}+\sqrt{3})^2}} & \frac{\sqrt{2}-\sqrt{3}}{\sqrt{1+(\sqrt{2}-\sqrt{3})^2}} \\ \frac{1}{\sqrt{1+(\sqrt{2}+\sqrt{3})^2}} & \frac{1}{\sqrt{1+(\sqrt{2}-\sqrt{3})^2}} \end{pmatrix},$$

$$V = \begin{pmatrix} \frac{v_1}{\|v_1\|} & \frac{v_2}{\|v_2\|} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

As for the diagonal matrix,

$$\Sigma \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{3+\sqrt{6}}{6}} & 0 \\ 0 & \sqrt{\frac{3-\sqrt{6}}{6}} \end{pmatrix}.$$

Finally, as we saw earlier, our Schmidt decomposition gives us

$$|\psi\rangle_{AB} = \sigma_1 \cdot u_1 \otimes v_1 + \sigma_2 \cdot u_2 \otimes v_2.$$

Problem 6

For a general quantum state $|\psi\rangle$, the number of nonzero constants (Schmidt coefficients) c_i in its Schmidt decomposition is called the "Schmidt number" for the state $|\psi\rangle$.

(a) Prove that a pure state $|\psi\rangle_{AB}$ of a two-qubit bipartite system is entangled if and only if its Schmidt number is greater than 1.

Solution. We will prove this by contraposition. That is, we will show that $|\psi\rangle_{AB}$ is a product state if and only if its Schmidt number is equal to 1.

" $\Longrightarrow$ " If $|\psi\rangle_{AB}$ is a product state, then it can be written in the form $|\psi\rangle_{AB} = c_1 |e_1\rangle \otimes |f_1\rangle$. Therefore, its Schmidt number is equal to 1.

" $\Longleftarrow$ " For $|\psi\rangle_{AB}$, using the Schmidt decomposition and given that Schmidt number is equal to 1, we know we can express the state in the form $|\psi\rangle_{AB} = c_1 |e_1\rangle \otimes |f_1\rangle$. Therefore, $|\psi\rangle_{AB}$ is a product state.

(b) Suppose that $|\psi_1\rangle$ and $|\psi_2\rangle$ are two states of a two-qubit bipartite system (with components A and B) having identical Schmidt coefficients. Show that there are unitary transformations U on system A and V on system B such that $|\psi_1\rangle = (U \otimes V) |\psi_2\rangle$.

Solution. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are two-qubit states, we can write them using the Schmidt decomposition with respect to some orthonormal bases $\{|e_1\rangle_A, |e_2\rangle_A\}$ and $\{|f_1\rangle_B, |f_2\rangle_B\}$ for $|\psi_1\rangle$, and $\{|e_1'\rangle_A, |e_2'\rangle_A\}$ and $\{|f_1'\rangle_B, |f_2'\rangle_B\}$ for $|\psi_2\rangle$. Given that the two states have identical Schmidt coefficients, they can be expressed as

$$\begin{aligned} |\psi_1\rangle &= c_1 |e_1\rangle \otimes |f_1\rangle + c_2 |e_2\rangle \otimes |f_2\rangle, \\ |\psi_2\rangle &= c_1 |e_1'\rangle \otimes |f_1'\rangle + c_2 |e_2'\rangle \otimes |f_2'\rangle. \end{aligned}$$

Defining U and V as

$$\begin{aligned} U &= |e_1\rangle \langle e_1'| + |e_2\rangle \langle e_2'|, \\ V &= |f_1\rangle \langle f_1'| + |f_2\rangle \langle f_2'|, \end{aligned}$$

notice that

$$\begin{aligned} (U \otimes V) |\psi_2\rangle &= \left[\left(|e_1\rangle \langle e_1'| + |e_2\rangle \langle e_2'|\right) \otimes \left(|f_1\rangle \langle f_1'| + |f_2\rangle \langle f_2'|\right)\right] \left(c_1 |e_1'\rangle \otimes |f_1'\rangle + c_2 |e_2'\rangle \otimes |f_2'\rangle\right) \\ &= c_1 \left(|e_1\rangle \langle e_1'| + |e_2\rangle \langle e_2'|\right) |e_1'\rangle \otimes \left(|f_1\rangle \langle f_1'| + |f_2\rangle \langle f_2'|\right) |f_1'\rangle \\ &\quad + c_2 \left(|e_1\rangle \langle e_1'| + |e_2\rangle \langle e_2'|\right) |e_2'\rangle \otimes \left(|f_1\rangle \langle f_1'| + |f_2\rangle \langle f_2'|\right) |f_2'\rangle \\ &= c_1 |e_1\rangle \otimes |f_1\rangle + c_2 |e_2\rangle \otimes |f_2\rangle \\ &= |\psi_1\rangle. \end{aligned}$$