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Tutorial 2: Solutions

Problem 1: Bit-flip channel

The bit-flip channel does nothing with probability p, and flips a ket $|0\rangle$ (respectively $|1\rangle$) to a ket $|1\rangle$ (respectively $|0\rangle$) with probability $1-p$. Its effect on a state ρ is given by

$$\Phi_{BF}[\rho] = p\rho + (1-p)X\rho X,$$

where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Compute the output state of the bit-flip channel when its input is a general qubit mixed state

$$\rho = \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix}.$$

Solution. We have

$$X\rho X = \begin{pmatrix} 1-a & c^* \\ c & a \end{pmatrix}.$$

Therefore,

$$\Phi_{BF}[\rho] = p\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + (1-p)\begin{pmatrix} 1-a & c^* \\ c & a \end{pmatrix}.$$

(b) What is the effect of applying the bit-flip channel with $p = 1/2$ to one of the two qubits of the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$?

Hint: Doing nothing on the first qubit and applying a bit-flip channel on the second qubit has the effect of doing nothing with probability p and applying $I \otimes X$ with probability $1-p$.

Solution. The effect of the bit-flip channel acting on the second qubit of a two-qubit state ρ is given by

$$\mathbb{1} \otimes \Phi_{BF}[\rho] = p\rho + (1-p)(I \otimes X)\rho(I \otimes X).$$

Notice also that a local bit flip acting on the Bell state $|\Phi^+\rangle$ gives

$$(I \otimes X)|\Phi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \equiv |\Psi^+\rangle.$$

Setting $p = 1/2$ and $\rho = |\Phi^+\rangle\langle\Phi^+|$,

$$\begin{aligned}
\mathbb{1} \otimes \Phi_{BF}[|\Phi^+\rangle\langle\Phi^+|] &= \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}(I \otimes X)|\Phi^+\rangle\langle\Phi^+|(I \otimes X) \\
&= \frac{1}{2}(|\Phi^+\rangle\langle\Phi^+| + |\Psi^+\rangle\langle\Psi^+|) \\
&= \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}(|{++}\rangle\langle{++}| + |{--}\rangle\langle{--}|).
\end{aligned}$$

Notice that the resulting state can be seen as a classical coin in the basis $\{|+\rangle, |-\rangle\}$. This channel degrades entanglement, but we can always use it to send 1 bit of classical information per use of the channel by choosing the correct basis (here, using the $\{|+\rangle, |-\rangle\}$ basis) irrespective of noise. It is interesting, then, that the classical capacity of this quantum channel is unaffected by noise.

Problem 2: Depolarizing channel

The depolarizing channel applies a bit-flip, phase-flip, or both, with probability $p/4$, and does nothing with probability $1 - 3p/4$. Its effect on a state ρ is given by

$$\Phi_{DE}[\rho] = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z),$$

where

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Compute the output state of the depolarizing channel when its input is a general qubit mixed state ρ.

Solution. If ρ is the general qubit state as before

$$\rho = \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix},$$

then it evolves through the depolarizing channel as

$$\Phi_{DE}[\rho] = \left(1 - \frac{3p}{4}\right)\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + \frac{p}{4}\begin{pmatrix} 2-a & -c \\ -c^* & 1+a \end{pmatrix}.$$

(b) Show that $\Phi_{DE}[\rho] = (1-p)\rho + p\mathbb{1}/2$, where $\mathbb{1}$ represents the identity matrix.

Solution. The expression found for $\Phi_{DE}[\rho]$ in the previous part can be simplified as

$$\begin{aligned}
\Phi_{DE}[\rho] &= \left(1 - \frac{3p}{4}\right)\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + \frac{p}{4}\begin{pmatrix} 2-a & -c \\ -c^* & 1+a \end{pmatrix} \\
&= \left(1 - p + \frac{p}{4}\right)\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + \frac{p}{4}\begin{pmatrix} 2-a & -c \\ -c^* & 1+a \end{pmatrix} \\
&= (1-p)\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + \frac{p}{4}\left[\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + \begin{pmatrix} 2-a & -c \\ -c^* & 1+a \end{pmatrix}\right] \\
&= (1-p)\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + \frac{p}{4}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
&= (1-p)\rho + p\frac{\mathbb{1}}{2}.
\end{aligned}$$

Problem 3: Amplitude damping channel

The Kraus representation of the damping channel is given by

$$\Phi_{DA}[\rho] = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger,$$

where

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

(a) Check that $\sum_k E_k^\dagger E_k = \mathbb{1}$.

Solution. We can show that

$$E_0^\dagger E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix}, \quad E_1^\dagger E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix},$$

meaning that we indeed have $E_0^\dagger E_0 + E_1^\dagger E_1 = \mathbb{1}$.

(b) Compute the output state of an amplitude damping channel when its input is the state

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution. We find that

$$E_0 \rho E_0^\dagger = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{1-\gamma} \\ \sqrt{1-\gamma} & 1-\gamma \end{pmatrix},$$
$$E_1 \rho E_1^\dagger = \frac{1}{2}\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix},$$

which lead to

$$\Phi_{DA}[\rho] = \frac{1}{2}\begin{pmatrix} 1+\gamma & \sqrt{1-\gamma} \\ \sqrt{1-\gamma} & 1-\gamma \end{pmatrix}.$$

We see that the off-diagonal entries decay to 0 with the parameter $\gamma \in [0,1]$, thus destroying coherence in the state. These off-diagonal terms vanish more quickly than the diagonal term for $|1\rangle\langle 1|$ due to their square root. In fact, the amplitude damping channel models energy relaxation of a system from an excited state $|1\rangle$ to its ground state $|0\rangle$, with γ the decay probability.

Problem 4: Freedom in operator-sum representation

Consider a quantum channel $\mathcal{L}$ defined by the operator sum representation

$$\mathcal{L}[\rho] = \sum_j E_j \rho E_j^\dagger,$$

with Kraus operators given by

$$E_0 = |0\rangle\langle 0|, \quad E_1 = |1\rangle\langle 1|.$$

Consider also a quantum channel $\mathcal{F}$ defined by the operator sum representation

$$\mathcal{F}[\rho] = \sum_j F_j \rho F_j^\dagger,$$

with Kraus operators given by

$$F_0 = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{\sqrt{2}}, \quad F_1 = \frac{|0\rangle\langle 0| - |1\rangle\langle 1|}{\sqrt{2}}.$$

(a) Given that the input state is a general qubit state

$$\rho = \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix},$$

show that the output state of the channel $\mathcal{L}$ is

$$\mathcal{L}[\rho] = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix},$$

i.e., the channel $\mathcal{L}$ destroys all coherence in the initial quantum state.

Solution. Writing the Kraus operators in matrix form gives

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The action of the channel $\mathcal{L}$ is then

$$\begin{aligned}
\mathcal{L}[\rho] &= \sum_j E_j \rho E_j^\dagger \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1-a \end{pmatrix} \\
&= \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix}.
\end{aligned}$$

(b) Compute the output state of channel $\mathcal{F}$ when its input is the same general state ρ as in the previous part.

Solution. Writing the Kraus operators in matrix form gives

$$F_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The action of the channel $\mathcal{F}$ is then

$$\begin{aligned}
\mathcal{F}[\rho] &= \sum_j F_j \rho F_j^\dagger \\
&= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a & -c \\ -c^* & 1-a \end{pmatrix} \\
&= \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix}.
\end{aligned}$$

(c) Comment on the relation between the channels $\mathcal{L}$ and $\mathcal{F}$. What can we infer about the uniqueness of sets of Kraus operators for quantum channels?

Solution. We can see from the previous two parts that $\mathcal{L}[\rho] = \mathcal{F}[\rho]$ for any general qubit state ρ. That is, the quantum channels $\mathcal{L}$ and $\mathcal{F}$ are identical, despite being represented using different sets of Kraus operators. Therefore, the Kraus operators used to form an operator sum representation are not uniquely determined by the quantum channel in general.

In our example, we can write

$$F_0 = \frac{E_0 + E_1}{\sqrt{2}}, \quad F_1 = \frac{E_0 - E_1}{\sqrt{2}}.$$

Considering the (unitary) Hadamard matrix H with elements (h_{ij}) (indexed starting at 0) defined by

$$H \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we can write these relations compactly as $F_i = \sum_j h_{ij} E_j$.

We remark here that, in fact, all sets of Kraus operators representing the same quantum channel are related by some unitary transformation. Precisely, let $\{E_j\}_j$ and $\{F_j\}_j$ be two sets of Kraus matrices representing the same quantum channel, then there exists a unitary matrix U with elements $(u_{ij})_{ij}$ such that $F_i = \sum_j u_{ij} E_j$.

This unitary equivalence can be thought of physically as implementing a quantum channel as two different, but equivalent, quantum circuits. For example, the Kraus operators given for channel $\mathcal{L}$ correspond to the circuit depicted in Fig. 1. Its effect on an input state ρ corresponds to applying a CNOT gate before performing a measurement in the computational basis on the ancilla and forgetting the outcome of this measurement.

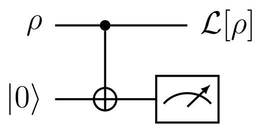

Figure 1: Channel $\mathcal{L}$. CNOT gate followed by a measurement in the computational basis.

The Kraus operators for channel $\mathcal{F}$, however, correspond to the circuit depicted in Fig. 2. Its effect on an input state ρ corresponds to applying a Hadamard gate on the ancilla, before using it to act on ρ with a controlled-Z gate, performing a measurement in the computational basis on the ancilla, and finally forgetting the outcome of this measurement.

One can show that these two circuits are equivalent up to a unitary before the measurement on the ancilla (specifically, a Hadamard gate before the ancilla measurement).

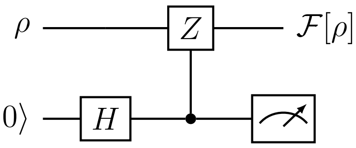

Figure 2: Channel $\mathcal{F}$. Hadamard gate on the ancilla followed by a controlled-Z gate.

Problem 5: Positive Operator-Valued Measurement

Consider the set of matrices

$$M_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |0\rangle\langle 0| , \quad M_2 = \frac{\sqrt{2}}{1+\sqrt{2}} |+\rangle\langle +| ,$$
$$M_3 = \frac{1-\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |0\rangle\langle 0| + \frac{1+\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |1\rangle\langle 1| - \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}} (|0\rangle\langle 1| + |1\rangle\langle 0|)$$

(a) i. Check that the completeness relation is satisfied, i.e., $\sum_j M_j = I$.

Solution. We need to check that $M_1 + M_2 + M_3 = I$. We have

$$\begin{aligned}
M_1 + M_2 + M_3 &= \frac{\sqrt{2}}{1+\sqrt{2}} |0\rangle\langle 0| + \frac{\sqrt{2}}{1+\sqrt{2}} |+\rangle\langle +| \\
&+ \frac{1-\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |0\rangle\langle 0| + \frac{1+\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |1\rangle\langle 1| - \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}} (|0\rangle\langle 1| + |1\rangle\langle 0|) \\
&= \frac{\sqrt{2}}{1+\sqrt{2}} |0\rangle\langle 0| + \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}} (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&+ \frac{1-\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |0\rangle\langle 0| + \frac{1+\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |1\rangle\langle 1| - \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}} (|0\rangle\langle 1| + |1\rangle\langle 0|) \\
&= \frac{\sqrt{2}+\frac{\sqrt{2}}{2}+1-\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |0\rangle\langle 0| + \frac{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |0\rangle\langle 1| \\
&+ \frac{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |1\rangle\langle 0| + \frac{\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |1\rangle\langle 1| \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \\
&= I,
\end{aligned}$$

which proves that the completeness relation is satisfied.

ii. It can be shown that M_1 and M_3 are positive semidefinite matrices. Show that M_2 also is, i.e., M_2 has no negative eigenvalues.

Solution. Using the fact that $\{|+\rangle, |-\rangle\}$ is an orthonormal basis of $\mathbb{C}^2$, it is easy to see that

$$M_2 |+\rangle = \frac{\sqrt{2}}{1+\sqrt{2}} |+\rangle , \quad M_2 |-\rangle = 0 |-\rangle .$$

We conclude that $\lambda_1 = 0$ and $\lambda_2 = \frac{\sqrt{2}}{1+\sqrt{2}}$ are the eigenvalues of M_2, which are both non-negative. Therefore, M_2 is a positive semidefinite matrix.

Solution. (alternative) The eigenvalues of M_2 are the roots $\lambda \in \mathbb{C}$ of its characteristic equation $\det(M_2 - \lambda I) = 0$. Let $a \equiv \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}}$. We can rewrite M_2 in matrix form as

$$\begin{aligned} M_2 &= \frac{\sqrt{2}}{1+\sqrt{2}} |+\rangle\langle+| \\ &= a(|0\rangle\langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle\langle 1|) \\ &= \begin{pmatrix} a & a \\ a & a \end{pmatrix} \end{aligned}$$

This yields

$$\begin{aligned} \det(M_2 - \lambda I) &= \det\left(\begin{pmatrix} a-\lambda & a \\ a & a-\lambda \end{pmatrix}\right) \\ &= (a-\lambda)^2 - a^2 \\ &= \lambda^2 - 2a\lambda \\ &= \lambda(\lambda - 2a) \end{aligned}$$

We conclude that $\lambda_1 = 0$ and $\lambda_2 = 2a = \frac{\sqrt{2}}{1+\sqrt{2}}$ are the eigenvalues of M_2 which are both non negative. Therefore, M_2 is a positive semidefinite matrix.

iii. Conclude on whether or not $\{M_1, M_2, M_3\}$ is a valid positive operator-valued measurement (POVM).

Solution. We have shown that $\{M_1, M_2, M_3\}$ are positive semidefinite matrices that satisfy the completeness relation $\sum_j M_j = I$; we can conclude that $\{M_1, M_2, M_3\}$ is a valid POVM on $\mathbb{C}^2$.

(b) Suppose that you were given a qubit by Alice. All you know is that she prepared it in one of two states:

$$|\Psi_1\rangle = |1\rangle, \quad |\Psi_2\rangle = |-\rangle.$$

i. Show that the probability of getting outcome 1 with measurement $\{M_1, M_2, M_3\}$ if you received state $|\Psi_1\rangle$ is 0.

Solution.

$$\begin{aligned} p_{1,\Psi_1} &= \mathrm{tr}[M_1 |\Psi_1\rangle\langle\Psi_1|] \\ &= \langle\Psi_1| M_1 |\Psi_1\rangle \\ &= \langle 1| \frac{\sqrt{2}}{1+\sqrt{2}} |0\rangle\langle 0| |1\rangle \\ &= 0, \end{aligned}$$

where we have used the orthogonality of $|0\rangle$ and $|1\rangle$.

ii. Show that the probability of getting outcome 2 with measurement $\{M_1, M_2, M_3\}$ if you received state $|\Psi_2\rangle$ is 0.

Solution.

$$\begin{aligned} p_{2,\Psi_2} &= \mathrm{tr}[M_2 |\Psi_2\rangle\langle\Psi_2|] \\ &= \langle\Psi_2| M_2 |\Psi_2\rangle \\ &= \langle -| \frac{\sqrt{2}}{1+\sqrt{2}} |+\rangle\langle +| |-\rangle \\ &= 0, \end{aligned}$$

where we have used the orthogonality of $|+\rangle$ and $|-\rangle$.

iii. Compute the probability of obtaining outcome 3

Solution. Using the completeness relation, we have

$$\begin{aligned} p_{3,\Psi_i} &= \mathrm{tr}[M_3 |\Psi_i\rangle\langle\Psi_i|] \\ &= \langle\Psi_i| M_3 |\Psi_i\rangle \\ &= \langle\Psi_i| (I - M_1 - M_2) |\Psi_i\rangle \\ &= 1 - \langle\Psi_i| M_1 |\Psi_i\rangle - \langle\Psi_i| M_2 |\Psi_i\rangle \end{aligned}$$

for $i \in \{1, 2\}$.

A. if you received $|\Psi_1\rangle$.

Solution.

$$\begin{aligned} p_{3,\Psi_1} &= 1 - \langle\Psi_1| M_1 |\Psi_1\rangle - \langle\Psi_1| M_2 |\Psi_1\rangle \\ &= 1 - 0 - \langle 1| \frac{\sqrt{2}}{1+\sqrt{2}} |+\rangle\langle +| |1\rangle \\ &= 1 - \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}} \end{aligned}$$

B. if you received $|\Psi_2\rangle$.

Solution.

$$\begin{aligned} p_{3,\Psi_2} &= 1 - \langle\Psi_2| M_1 |\Psi_2\rangle - \langle\Psi_2| M_2 |\Psi_2\rangle \\ &= 1 - \langle -| \frac{\sqrt{2}}{1+\sqrt{2}} |0\rangle\langle 0| |-\rangle - 0 \\ &= 1 - \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}} \end{aligned}$$

iv. Discuss for each measurement outcome if you can infer something about the qubit prepared by Alice.

Solution. From previous answers, we deduce that

- if outcome 1 is obtained, then Alice sent $|\Psi_2\rangle$.
- if outcome 2 is obtained, then Alice sent $|\Psi_1\rangle$.
- if outcome 3 is obtained, then nothing can be inferred on the state sent by Alice.