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# Tutorial 1: Solutions

QCS 2023–24
January 24, 2024

## Problem 1: Quantum states

(a) Consider the quantum states $|v_1\rangle = \frac{1}{2}\begin{pmatrix}1+i\\1-i\end{pmatrix}$, $|v_2\rangle = \frac{1}{2}\begin{pmatrix}1-i\\1+i\end{pmatrix}$, and $|v_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

i. Write $\langle v_1|$ and $\langle v_2|$ in vector notation.
*Solution.* For any state vector represented by a column vector, the corresponding element of the dual space is represented by its conjugate transpose. In this case,

$$\langle v_1| = \frac{1}{2}\begin{pmatrix}1-i & 1+i\end{pmatrix}, \quad \langle v_2| = \frac{1}{2}\begin{pmatrix}1+i & 1-i\end{pmatrix}.$$

ii. Show that both $|v_1\rangle$ and $|v_2\rangle$ are normalised, i.e. $\sqrt{\langle v_1|v_1\rangle} = \sqrt{\langle v_2|v_2\rangle} = 1$.
*Solution.* For $|v_1\rangle$,

$$\begin{aligned}\langle v_1|v_1\rangle &= \frac{1}{4}\begin{pmatrix}1-i & 1+i\end{pmatrix}\begin{pmatrix}1+i\\1-i\end{pmatrix}\\ &= \frac{1}{4}[(1-i)(1+i)+(1+i)(1-i)]\\ &= \frac{1}{4}(2+2) = 1.\end{aligned}$$

Similarly, for $|v_2\rangle$,

$$\begin{aligned}\langle v_2|v_2\rangle &= \frac{1}{4}\begin{pmatrix}1+i & 1-i\end{pmatrix}\begin{pmatrix}1-i\\1+i\end{pmatrix}\\ &= \frac{1}{4}[(1+i)(1-i)+(1-i)(1+i)]\\ &= \frac{1}{4}(2+2) = 1.\end{aligned}$$

iii. Calculate the inner products $\langle v_1|v_2\rangle$ and $\langle v_3|v_1\rangle$. Are $|v_1\rangle$ and $|v_2\rangle$ orthogonal?
*Solution.* For the first inner product,

$$\begin{aligned}\langle v_1|v_2\rangle &= \frac{1}{4}\begin{pmatrix}1-i & 1+i\end{pmatrix}\begin{pmatrix}1-i\\1+i\end{pmatrix}\\ &= \frac{1}{4}[(1-i)(1-i)+(1+i)(1+i)]\\ &= \frac{1}{4}[(1-2i-1)+(1+2i-1)] = 0.\end{aligned}$$

For the second inner product,

$$\langle v_3|v_1\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1+i\\1-i\end{pmatrix} = \frac{1}{2\sqrt{2}}(1+i+1-i) = \frac{1}{\sqrt{2}}.$$

iv. Show that the set $\{|v_1\rangle, |v_2\rangle\}$ satisfies all the conditions of an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$.

*Solution.* The properties which must be satisfied are the following.

- The vectors must be orthogonal, which was already shown.
- The vectors must be normalised to unit length, which was already shown.
- The number of vectors must be the same as the dimension of the Hilbert space, which is true since there are 2 vectors and the Hilbert space is $\mathbb{C}^2$.

v. Write $|v_3\rangle$ as a linear combination of $|v_1\rangle$ and $|v_2\rangle$.

*Solution.* We want to find constants $a, b \in \mathbb{C}$ such that $|v_3\rangle = a|v_1\rangle + b|v_2\rangle$. Writing this in vector notation gives

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a+b+(a-b)i\\a+b-(a-b)i\end{pmatrix},$$

and so we must have $a = b = \frac{1}{\sqrt{2}}$.

(b) A general state can be represented by the superposition

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle,$$

where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, and $\{|0\rangle, |1\rangle\}$ is the computational basis.

i. Prove that $|\psi\rangle$ is normalised.

*Solution.* We must show that $\||\psi\rangle\| = 1$, and since $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$, this is equivalent to showing $\langle\psi|\psi\rangle = 1$. Using the trigonometric identity $\cos^2 x + \sin^2 x = 1$ for all $x$, this is

$$\langle\psi|\psi\rangle = \begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix} = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1.$$

ii. Find the values of $\theta$ and $\varphi$ such that

A. $|\psi\rangle = |v_3\rangle$,

*Solution.* First notice that by equating the coefficients of $|0\rangle$, we must have $1/\sqrt{2} = \cos(\theta/2)$, and thus $\theta = \pi/2$ since $\theta \in [0, \pi]$ is required. To find $\varphi$, equating the coefficients of $|1\rangle$ we find

$$\frac{1}{\sqrt{2}} = e^{i\varphi}\sin\frac{\theta}{2} = \frac{e^{i\varphi}}{\sqrt{2}}$$

and therefore $\varphi = 0$, since we require $\varphi \in [0, 2\pi)$.

B. $|\psi\rangle = e^{-i\pi/4}|v_1\rangle$.

*Solution.* First we can evaluate

$$\begin{aligned} e^{-i\pi/4}|v_1\rangle = \frac{1}{2}e^{-i\pi/4}\begin{pmatrix}1+i\\1-i\end{pmatrix} &= \frac{1}{2}e^{-i\pi/4}\begin{pmatrix}1+i\\1-i\end{pmatrix} \\ &= \frac{\sqrt{2}}{2}e^{-i\pi/4}\begin{pmatrix}e^{i\pi/4}\\e^{-i\pi/4}\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}. \end{aligned}$$

Thus, by equating $1/\sqrt{2} = \cos(\theta/2)$, we see that $\theta = \pi/2$. Now by equating the other coefficients, we must have

$$\frac{-i}{\sqrt{2}} = e^{i\varphi} \sin\frac{\theta}{2} = \frac{e^{i\varphi}}{\sqrt{2}}$$

and therefore $\varphi = 3\pi/2$. Note that since normalised state vectors are considered to be equivalent up to arbitrary multiplication by a unit complex number (also called a "phase factor"), the state $|\psi\rangle$ found with $\theta = \pi/2$ and $\varphi = 3\pi/2$ in fact also represents the state $|v_1\rangle$.

## Problem 2: Quantum operations

Some important linear operators in quantum computing are the three *Pauli* operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Prove the following properties of the Pauli operators.

i. They are self-adjoint, i.e. $X = X^\dagger$, $Y = Y^\dagger$, and $Z = Z^\dagger$.

*Solution.* In matrix notation, the adjoint of an operator $O$ acts as the conjugate transpose $O^\dagger = (O^*)^\mathsf{T}$. Thus, we see

$$X^\dagger = \begin{pmatrix} 0^* & 1^* \\ 1^* & 0^* \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X,$$
$$Y^\dagger = \begin{pmatrix} 0^* & i^* \\ (-i)^* & 0^* \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y,$$
$$Z^\dagger = \begin{pmatrix} 1^* & 0^* \\ 0^* & (-1)^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z.$$

ii. They are self-inverse, i.e. $X^2 = I$, $Y^2 = I$, and $Z^2 = I$, where $I$ is the identity operator.

*Solution.* By direct calculation,

$$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$
$$Y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$
$$Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

iii. The operators $Y$ and $Z$ anticommute, i.e. $YZ = -ZY$.

*Solution.* We evaluate the *anticommutator* denoted using the notation $\{A, B\} \equiv AB + BA$. That $YZ = -ZY$ is clearly equivalent to $\{Y, Z\} = 0$. By direct calculation,

$$\begin{aligned} YZ + ZY &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

(b) Consider a linear operator defined by

$$U \equiv \frac{Y + Z}{\sqrt{2}}.$$

Using properties from the previous part, show that $U$ is unitary, i.e. $U^\dagger U = UU^\dagger = I$.

*Solution.* Since we already showed $Y = Y^\dagger$ and $Z = Z^\dagger$, we have

$$U^\dagger = \frac{Y^\dagger + Z^\dagger}{\sqrt{2}} = \frac{Y + Z}{\sqrt{2}} = U.$$

Therefore, $U^\dagger U = UU^\dagger = U^2$ and so showing that $U$ is unitary (that $U^\dagger U = UU^\dagger = I$) is equivalent to showing $U^2 = I$. Finally, we can now use the properties $Y^2 = I$, $Z^2 = I$, and $YZ = -ZY$ showed previously to see

$$U^2 = \left(\frac{Y + Z}{\sqrt{2}}\right)^2 = \frac{Y^2 + YZ + ZY + Z^2}{2} = \frac{I - ZY + ZY + I}{2} = I.$$

(c) Calculate the action of the operator $U$ on the vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |{+i}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |{-i}\rangle = \frac{-1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

*Solution.* Writing $U$ in matrix notation,

$$U = \frac{Y + Z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}.$$

Applying this to each of the given column vectors,

$$\begin{aligned} U|0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = |{+i}\rangle, \\ U|1\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = |{-i}\rangle, \\ U|{+i}\rangle &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \\ U|{-i}\rangle &= -\frac{1}{2} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle. \end{aligned}$$

# Problem 3: Tensor product

(a) Consider the quantum state

$$|\psi\rangle = \frac{\sqrt{5}}{5}|0\rangle + \frac{2\sqrt{5}}{5}|1\rangle .$$

i. Express $|\psi\rangle^{\otimes 2}$ in Dirac notation, where $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$.

*Solution.* Using the bilinear property of tensor products, we can substitute the given state to yield

$$\begin{aligned}
|\psi\rangle^{\otimes 2} &= |\psi\rangle \otimes \left(\frac{\sqrt{5}}{5}|0\rangle + \frac{2\sqrt{5}}{5}|1\rangle\right) \\
&= \frac{\sqrt{5}}{5}|\psi\rangle \otimes |0\rangle + \frac{2\sqrt{5}}{5}|\psi\rangle \otimes |1\rangle \\
&= \left(\frac{1}{5}|0\rangle + \frac{2}{5}|1\rangle\right) \otimes |0\rangle + \left(\frac{2}{5}|0\rangle + \frac{4}{5}|1\rangle\right) \otimes |1\rangle \\
&= \frac{1}{5}|0\rangle \otimes |0\rangle + \frac{2}{5}|1\rangle \otimes |0\rangle + \frac{2}{5}|0\rangle \otimes |1\rangle + \frac{4}{5}|1\rangle \otimes |1\rangle \\
&= \frac{1}{5}|00\rangle + \frac{2}{5}|10\rangle + \frac{2}{5}|01\rangle + \frac{4}{5}|11\rangle ,
\end{aligned}$$

where the final equality is simply the use of a compact notation for tensor products.

ii. Express $|+\rangle|+\rangle|-\rangle$ in Dirac notation, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

*Solution.* Noticing the shorthand notation $|+\rangle|+\rangle|-\rangle \equiv |+\rangle \otimes |+\rangle \otimes |-\rangle$, we can again use the bilinear property of tensor products to write

$$\begin{aligned}
|+\rangle|+\rangle|-\rangle &= \frac{\sqrt{2}}{4}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) \\
&= \frac{\sqrt{2}}{4}(|0\rangle|0\rangle + |1\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|1\rangle) \otimes (|0\rangle - |1\rangle) \\
&= \frac{\sqrt{2}}{4}(|000\rangle + |100\rangle + |010\rangle + |110\rangle - |001\rangle - |101\rangle - |011\rangle - |111\rangle).
\end{aligned}$$

(b) Consider the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

i. Express the tensor products $X \otimes I$ and $I \otimes X$ as two $4 \times 4$ matrices.

*Solution.* The tensor product of operators written in matrix notation is called the “Kronecker product” operation. Evaluating the first tensor product,

$$X \otimes I = \begin{pmatrix} 0I & 1I \\ 1I & 0I \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now, evaluating the second tensor product,

$$I \otimes X = \begin{pmatrix} 1X & 0X \\ 0X & 1X \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

ii. Express the tensor products $X \otimes Z$ and $Z \otimes Y$ as matrices, then calculate the matrix multiplication $(X \otimes Z)(Z \otimes Y)$.

*Solution.* Evaluating the first tensor product,

$$X \otimes Z = \begin{pmatrix} 0Z & 1Z \\ 1Z & 0Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Evaluating the second tensor product,

$$Z \otimes Y = \begin{pmatrix} 1Y & 0Y \\ 0Y & -1Y \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

Finally, evaluating the matrix multiplication,

$$(X \otimes Z)(Z \otimes Y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

iii. Calculate the matrices $XZ$ and $ZY$, and hence verify the special case

$$(X \otimes Z)(Z \otimes Y) = (XZ) \otimes (ZY)$$

of the more general identity $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

*Solution.* Evaluating the first matrix,

$$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Evaluating the second matrix,

$$ZY = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Now, evaluating the tensor product $(XZ) \otimes (ZY)$ gives

$$(XZ) \otimes (ZY) = \begin{pmatrix} 0ZY & -1ZY \\ 1ZY & 0ZY \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = (X \otimes Z)(Z \otimes Y)$$

as we had to verify.

(c) Prove that if $A$ and $B$ are projection operators then $A \otimes B$ is a projection operator.

*Solution.* An operator $O$ is called a *projection* operator if and only if $O^2 = O$. Thus we have to prove that $(A \otimes B)^2 = (A \otimes B)$. Using the general property of operator tensor products that $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, and then that $A^2 = A$ and $B^2 = B$ since $A$ and $B$ are projection operators,

$$(A \otimes B)^2 = (A \otimes B)(A \otimes B) = (AA) \otimes (BB) = A^2 \otimes B^2 = A \otimes B.$$

(d) Prove that if $A$ and $B$ are unitary operators then $A \otimes B$ is a unitary operator. You may use the property that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

*Solution.* To show that $A \otimes B$ is a unitary operator, we must show that

$$(A \otimes B)^\dagger (A \otimes B) = (A \otimes B)(A \otimes B)^\dagger = I.$$

Since $A$ and $B$ are unitary, we have that $A^\dagger A = AA^\dagger = I$ and $B^\dagger B = BB^\dagger = I$. Using also the property given in the question, we see the first equality required that

$$(A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B) = (A^\dagger A) \otimes (B^\dagger B) = I \otimes I = I.$$

The other equality required follows similarly:

$$(A \otimes B)(A \otimes B)^\dagger = (A \otimes B)(A^\dagger \otimes B^\dagger) = (AA^\dagger) \otimes (BB^\dagger) = I \otimes I = I.$$

# Problem 4: Quantum measurements

(a) For each the following states, calculate the probabilities $p_0$ and $p_1$ of obtaining outcomes 0 and 1 from a measurement in the computational basis $\{|0\rangle, |1\rangle\}$, and the probabilities $p_+$ and $p_-$ of obtaining outcomes $+$ and $-$ from a measurement in the basis $\{|+\rangle, |-\rangle\}$.

*Solution.* Recall that when measuring a state $|\psi\rangle$ in an orthonormal basis $\{|v_i\rangle\}_i$, the probability of obtaining outcome $i$ is given by $p_i = \|P_i |\psi\rangle\|^2 = |\langle v_i|\psi\rangle|^2$, where $P_i = |v_i\rangle \langle v_i|$ is the projector onto the vector $|v_i\rangle$.

For a measurement in the computational basis $\{|0\rangle, |1\rangle\}$, the projector onto $|0\rangle$ is given by $P_0 = |0\rangle \langle 0|$ and the projector onto $|1\rangle$ is given by $P_1 = |1\rangle \langle 1|$. Similarly, for a measurement in the basis $\{|+\rangle, |-\rangle\}$, the projector onto $|+\rangle$ is given by $P_+ = |+\rangle \langle +|$ and the projector onto $|-\rangle$ is given by $P_- = |-\rangle \langle -|$.

i. $|\psi_1\rangle = |1\rangle$.

*Solution.* We evaluate the desired probabilities as follows.

$$\begin{aligned} p_0 &= \|P_0 |\psi_1\rangle\|^2 = |\langle 0|\psi_1\rangle|^2 = |\langle 0|1\rangle|^2 = 0, \\ p_1 &= \|P_1 |\psi_1\rangle\|^2 = |\langle 1|\psi_1\rangle|^2 = |\langle 1|1\rangle|^2 = 1, \\ p_+ &= \|P_+ |\psi_1\rangle\|^2 = |\langle +|\psi_1\rangle|^2 = |\langle +|1\rangle|^2 = \frac{1}{2}, \\ p_- &= \|P_- |\psi_1\rangle\|^2 = |\langle -|\psi_1\rangle|^2 = |\langle -|1\rangle|^2 = \frac{1}{2}. \end{aligned}$$

ii. $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
*Solution.* We evaluate the desired probabilities as follows.

$$\begin{aligned}
p_0 &= \|P_0 |\psi_2\rangle\|^2 = |\langle 0|\psi_2\rangle|^2 = \frac{1}{2}|\langle 0|0\rangle + \langle 0|1\rangle|^2 = \frac{1}{2},\\
p_1 &= \|P_1 |\psi_2\rangle\|^2 = |\langle 1|\psi_2\rangle|^2 = \frac{1}{2}|\langle 1|0\rangle + \langle 1|1\rangle|^2 = \frac{1}{2},\\
p_+ &= \|P_+ |\psi_2\rangle\|^2 = |\langle +|\psi_2\rangle|^2 = \frac{1}{2}|\langle +|0\rangle + \langle +|1\rangle|^2\\
&= \frac{1}{4}|\langle 0|0\rangle + \langle 1|0\rangle + \langle 0|1\rangle + \langle 1|1\rangle|^2 = \frac{1}{4}|1+0+0+1|^2 = 1,\\
p_- &= \|P_- |\psi_2\rangle\|^2 = |\langle -|\psi_2\rangle|^2 = \frac{1}{2}|\langle -|0\rangle + \langle -|1\rangle|^2\\
&= \frac{1}{4}|\langle 0|0\rangle - \langle 1|0\rangle + \langle 0|1\rangle - \langle 1|1\rangle|^2 = \frac{1}{4}|1-0+0-1|^2 = 0.
\end{aligned}$$

iii. $|\psi_3\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, where $\alpha \neq 0$.
*Solution.* We evaluate the desired probabilities as follows.

$$\begin{aligned}
p_0 &= \|P_0 |\psi_3\rangle\|^2 = |\langle 0|\psi_3\rangle|^2 = |\alpha\,\langle 0|0\rangle + \beta\,\langle 0|1\rangle|^2 = |\alpha|^2,\\
p_1 &= \|P_1 |\psi_3\rangle\|^2 = |\langle 1|\psi_3\rangle|^2 = |\alpha\,\langle 1|0\rangle + \beta\,\langle 1|1\rangle|^2 = |\beta|^2,\\
p_+ &= \|P_+ |\psi_3\rangle\|^2 = |\langle +|\psi_3\rangle|^2 = |\alpha\,\langle +|0\rangle + \beta\,\langle +|1\rangle|^2\\
&= \frac{1}{2}|\alpha\,\langle 0|0\rangle + \alpha\,\langle 1|0\rangle + \beta\,\langle 0|1\rangle + \beta\,\langle 1|1\rangle|^2 = \frac{1}{2}|\alpha+\beta|^2,\\
p_- &= \|P_- |\psi_3\rangle\|^2 = |\langle -|\psi_3\rangle|^2 = |\alpha\,\langle -|0\rangle + \beta\,\langle -|1\rangle|^2\\
&= \frac{1}{2}|\alpha\,\langle 0|0\rangle - \alpha\,\langle 1|0\rangle + \beta\,\langle 0|1\rangle - \beta\,\langle 1|1\rangle|^2 = \frac{1}{2}|\alpha-\beta|^2.
\end{aligned}$$

(b) If the outcome of a measurement in the computational basis was 0, which of $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ were possible states of the system immediately before the measurement took place?

*Solution.* An outcome of 0 is possible if and only if the probability of it occurring is nonzero. Thus, we look for states that satisfy $p_0 \neq 0$. Of the three states considered, we can rule out $|\psi_1\rangle$, since we calculated that outcome 0 occurs with zero probability. The other two states, $|\psi_2\rangle$ and $|\psi_3\rangle$, are possible states, since for both we previously calculated $p_0 > 0$. For the state $|\psi_3\rangle$, this follows since it is given that $\alpha \neq 0$.

## Problem 5: Mixed states

(a) Consider the pure state formed by equal superposition of $|0\rangle$ and $|1\rangle$,

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and the *maximally mixed* state whose density matrix is given by

$$\sigma = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}.$$

i. Show that the density matrix $\rho = |\psi\rangle\langle\psi|$ of the pure state $|\psi\rangle$ is given by

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).$$

*Solution.* We expand the outer product

$$\rho = |\psi\rangle\langle\psi| = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right) = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).$$

ii. For the two mixed states $\rho$ and $\sigma$, calculate the probabilities of obtaining outcomes 0 and 1 from a measurement in the computational basis $\{|0\rangle, |1\rangle\}$.

*Solution.* Recall that when measuring a mixed state $\rho$ using the set of projectors $\{P_j\}_{j=1}^m$, the probability of obtaining outcome $k$ is given by $p_k^\rho = \text{tr}(\rho P_k)$. If $\{|v_j\rangle\}_{j=1}^n$ is an orthonormal basis, and the projector $P_k = |v_k\rangle\langle v_k|$, then this expression simplifies as

$$\begin{aligned} p_k^\rho &= \text{tr}(\rho P_k) = \text{tr}(\rho\,|v_k\rangle\langle v_k|) \\ &= \sum_{j=1}^n \langle v_j|(\rho\,|v_k\rangle\langle v_k|)|v_j\rangle \\ &= \sum_{j=1}^n \delta_{jk}\,\langle v_j|\rho|v_k\rangle = \langle v_k|\rho|v_k\rangle\,, \end{aligned}$$

where $\delta_{jk}$ is the Kronecker delta of $j$ and $k$.

The projectors for outcomes 0 and 1 are given by $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ respectively. For the the state $\rho$, the probabilities of obtaining each outcome are given by

$$\begin{aligned} p_0^\rho &= \langle 0|\rho|0\rangle = \frac{1}{2}(\langle 0|0\rangle\langle 0|0\rangle + \langle 0|0\rangle\langle 1|0\rangle + \langle 0|1\rangle\langle 0|0\rangle + \langle 0|1\rangle\langle 1|0\rangle) \\ &= \frac{1}{2}(1+0+0+0) = \frac{1}{2}, \\ p_1^\rho &= \langle 1|\rho|1\rangle = \frac{1}{2}(\langle 1|0\rangle\langle 0|1\rangle + \langle 1|0\rangle\langle 1|1\rangle + \langle 1|1\rangle\langle 0|1\rangle + \langle 1|1\rangle\langle 1|1\rangle) \\ &= \frac{1}{2}(0+0+0+1) = \frac{1}{2}. \end{aligned}$$

For the the state $\sigma$, the probabilities of obtaining each outcome are given by

$$\begin{aligned} p_0^\sigma &= \langle 0|\sigma|0\rangle = \frac{1}{2}(\langle 0|0\rangle\langle 0|0\rangle + \langle 0|1\rangle\langle 1|0\rangle) = \frac{1}{2}(1+0) = \frac{1}{2}, \\ p_1^\sigma &= \langle 1|\sigma|1\rangle = \frac{1}{2}(\langle 1|0\rangle\langle 0|1\rangle + \langle 1|1\rangle\langle 1|1\rangle) = \frac{1}{2}(0+1) = \frac{1}{2}. \end{aligned}$$

To summarise, $p_0^\rho = p_1^\rho = p_0^\sigma = p_1^\sigma = 1/2$.

iii. For the two mixed states $\rho$ and $\sigma$, calculate the probabilities of obtaining outcomes $+$ and $-$ from a measurement in the basis $\{|+\rangle, |-\rangle\}$.

*Solution.* The projectors for outcomes + and − are given by $P_+ = |+\rangle\langle+|$ and $P_- = |-\rangle\langle-|$ respectively. For the the state $\rho$, the probabilities of obtaining each outcome are given by

$$\begin{aligned} p_+^\rho = \langle+|\rho|+\rangle &= \frac{1}{2}(\langle+|0\rangle\langle0|+\rangle + \langle+|0\rangle\langle1|+\rangle + \langle+|1\rangle\langle0|+\rangle + \langle+|1\rangle\langle1|+\rangle) \\ &= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right) = 1, \\ p_-^\rho = \langle-|\rho|-\rangle &= \frac{1}{2}(\langle-|0\rangle\langle0|-\rangle + \langle-|0\rangle\langle1|-\rangle + \langle-|1\rangle\langle0|-\rangle + \langle-|1\rangle\langle1|-\rangle) \\ &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{2}\right) = 0. \end{aligned}$$

For the the state $\sigma$, the probabilities of obtaining each outcome are given by

$$\begin{aligned} p_+^\sigma = \langle+|\sigma|+\rangle &= \frac{1}{2}(\langle+|0\rangle\langle0|+\rangle + \langle+|1\rangle\langle1|+\rangle) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{2}, \\ p_-^\sigma = \langle-|\sigma|-\rangle &= \frac{1}{2}(\langle-|0\rangle\langle0|-\rangle + \langle-|1\rangle\langle1|-\rangle) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

To summarise, $p_+^\rho = 1$, $p_-^\rho = 0$, and $p_+^\sigma = p_-^\sigma = 1/2$.

iv. Comment on the distinguishability of the two states $|\psi\rangle$ and $\sigma$ with respect to the two measurement bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$.

*Solution.* When measured in the basis $\{|0\rangle, |1\rangle\}$, the equal superposition state $|\psi\rangle$ cannot be distinguished from the maximally mixed state $\sigma$. This is because the probabilities of all outcomes are the same whether the system is in state $|\psi\rangle$ or state $\sigma$; in both cases the probabilities of each outcome is 1/2. When measured in the basis $\{|+\rangle, |-\rangle\}$, however, we can distinguish the two states. The state $|\psi\rangle$ would result in outcomes distributed with probabilities $p_+^\rho = 1$ and $p_1^\rho = 0$, while the state $\sigma$ would result in outcomes with a different probability distribution $p_+^\sigma = p_-^\sigma = 1/2$.

(b) Recall that the density matrix $\rho$ for a statistical ensemble $\{(p_1, |\psi_1\rangle), \ldots, (p_n, |\psi_n\rangle)\}$ in which each pure state $|\psi_j\rangle$ occurs with probability $p_j$ is defined by

$$\rho = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j| .$$

i. Calculate the density matrix for the ensemble $\left\{\left(\frac{2}{3}, |0\rangle\right), \left(\frac{1}{3}, |1\rangle\right)\right\}$.

*Solution.* Let us denote by $\rho_1$ the density matrix of this ensemble. Then

$$\rho_1 = \frac{2}{3}|0\rangle\langle0| + \frac{1}{3}|1\rangle\langle1| .$$

We can also write this in matrix form as

$$\rho_1 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

ii. Calculate the density matrix for the ensemble $\left\{\left(\frac{1}{3}, |0\rangle\right), \left(\frac{1}{3}, |+\rangle\right), \left(\frac{1}{3}, |-\rangle\right)\right\}$.

*Solution.* Let us denote by $\rho_2$ the density matrix of this ensemble. Then

$$\begin{aligned}\rho_2 &= \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -| \\ &= \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \\ &= \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|.\end{aligned}$$

We can also write this in matrix form as

$$\rho_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

iii. Does there exist a measurement allowing an experimenter to distinguish between these two ensembles? Justify your answer.

*Solution.* There exists no measurement which can distinguish between the two ensembles. This follows from the fact that the density matrices for both ensembles are identical; $\rho_1 = \rho_2$. That is, the two ensembles give rise to the same (mixed) state. Since in both cases the system is in the same state, the probability distribution of outcomes for any measurement will be the the same in both cases. An experimenter may only rely on these outcome statistics to determine information about the system, therefore the two ensembles are said to be indistinguishable from one another.