Problem 1: Quantum states

(a) Consider the quantum states $|v_1\rangle = \frac{1}{2} \left( \frac{1+i}{1-i} \right)$, $|v_2\rangle = \frac{1}{2} \left( \frac{1-i}{1+i} \right)$, and $|v_3\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)$.

i. Write $\langle v_1 |$ and $\langle v_2 |$ in vector notation.
ii. Show that both $|v_1\rangle$ and $|v_2\rangle$ are normalised, i.e. $\sqrt{\langle v_1 | v_1 \rangle} = \sqrt{\langle v_2 | v_2 \rangle} = 1$.
iii. Calculate the inner products $\langle v_1 | v_2 \rangle$ and $\langle v_2 | v_1 \rangle$. Are $|v_1\rangle$ and $|v_2\rangle$ orthogonal?
iv. Show that the set $\{ |v_1\rangle, |v_2\rangle \}$ satisfies all the conditions of an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$.
v. Write $|v_3\rangle$ as a linear combination of $|v_1\rangle$ and $|v_2\rangle$.

(b) A general state can be represented by the superposition

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle,$$

where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, and $\{ |0\rangle, |1\rangle \}$ is the computational basis.

i. Prove that $|\psi\rangle$ is normalised.
ii. Find the values of $\theta$ and $\varphi$ such that

A. $|\psi\rangle = |v_3\rangle$,
B. $|\psi\rangle = e^{-i\pi/4} |v_1\rangle$.

Problem 2: Quantum operations

Some important linear operators in quantum computing are the three Pauli operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(a) Prove the following properties of the Pauli operators.

i. They are self-adjoint, i.e. $X = X^\dagger$, $Y = Y^\dagger$, and $Z = Z^\dagger$.
ii. They are self-inverse, i.e. $X^2 = I$, $Y^2 = I$, and $Z^2 = I$, where $I$ is the identity operator.
iii. The operators $Y$ and $Z$ anticommute, i.e. $YZ = -ZY$.

(b) Consider a linear operator defined by

$$U = \frac{Y + Z}{\sqrt{2}}.$$  

Using properties from the previous part, show that $U$ is unitary, i.e. $U^\dagger U = UU^\dagger = I$.

(c) Calculate the action of the operator $U$ on the vectors

$$|0\rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad |1\rangle = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad |+\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ i \end{array} \right), \quad |-\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \end{array} \right).$$
Problem 3: Tensor product

(a) Consider the quantum state

$$|\psi\rangle = \frac{\sqrt{5}}{5} |0\rangle + \frac{2\sqrt{5}}{5} |1\rangle.$$  

i. Express $|\psi\rangle^\otimes 2$ in Dirac notation, where $|\psi\rangle^\otimes 2 = |\psi\rangle \otimes |\psi\rangle$.

ii. Express $|+\rangle |+\rangle |-\rangle$ in Dirac notation, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

(b) Consider the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

i. Express the tensor products $X \otimes I$ and $I \otimes X$ as two $4 \times 4$ matrices.

ii. Express the tensor products $X \otimes Z$ and $Z \otimes Y$ as matrices, then calculate the matrix multiplication $(X \otimes Z)(Z \otimes Y)$.

iii. Calculate the matrices $XZ$ and $ZY$, and hence verify the special case

$$(X \otimes Z)(Z \otimes Y) = (XZ) \otimes (ZY)$$

of the more general identity $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

(c) Prove that if $A$ and $B$ are projection operators then $A \otimes B$ is a projection operator.

(d) Prove that if $A$ and $B$ are unitary operators then $A \otimes B$ is a unitary operator. You may use the property that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Problem 4: Quantum measurements

(a) For each of the following states, calculate the probabilities $p_0$ and $p_1$ of obtaining outcomes 0 and 1 from a measurement in the computational basis $\{|0\rangle, |1\rangle\}$, and the probabilities $p_+\!$ and $p_-$ of obtaining outcomes $+$ and $-$ from a measurement in the basis $\{|+\rangle, |-\rangle\}$.

i. $|\psi_1\rangle = |1\rangle$.

ii. $|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$.

iii. $|\psi_3\rangle = \alpha |0\rangle + \beta |1\rangle$, where $\alpha \neq 0$.

(b) If the outcome of a measurement in the computational basis was 0, which of $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ were possible states of the system immediately before the measurement took place?
Problem 5: Mixed states

(a) Consider the pure state formed by equal superposition of $|0\rangle$ and $|1\rangle$,

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and the maximally mixed state whose density matrix is given by

$$\sigma = \frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2}.$$

i. Show that the density matrix $\rho = |\psi\rangle \langle \psi|$ of the pure state $|\psi\rangle$ is given by

$$\rho = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1| + |1\rangle \langle 0| + |0\rangle \langle 1|).$$

ii. For the two mixed states $\rho$ and $\sigma$, calculate the probabilities of obtaining outcomes 0 and 1 from a measurement in the computational basis $\{|0\rangle, |1\rangle\}$.

iii. For the two mixed states $\rho$ and $\sigma$, calculate the probabilities of obtaining outcomes + and − from a measurement in the basis $\{|+\rangle, |-\rangle\}$.

iv. Comment on the distinguishability of the two states $|\psi\rangle$ and $\sigma$ with respect to the two measurement bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$.

(b) Recall that the density matrix $\rho$ for a statistical ensemble $\{(p_1, |\psi_1\rangle), \ldots, (p_n, |\psi_n\rangle)\}$ in which each pure state $|\psi_j\rangle$ occurs with probability $p_j$ is defined by

$$\rho = \sum_{j=1}^{n} p_j |\psi_j\rangle \langle \psi_j|.$$

i. Calculate the density matrix for the ensemble $\{(\frac{1}{3}, |0\rangle), (\frac{1}{3}, |1\rangle)\}$.

ii. Calculate the density matrix for the ensemble $\{(\frac{1}{3}, |0\rangle), (\frac{1}{3}, |+\rangle), (\frac{1}{3}, |-\rangle)\}$.

iii. Does there exist a measurement allowing an experimenter to distinguish between these two ensembles? Justify your answer.