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Tutorial 1

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Problem 1: Quantum states

(a) Consider the quantum states $|v_1\rangle = \frac{1}{2}\begin{pmatrix}1+i\\1-i\end{pmatrix}$, $|v_2\rangle = \frac{1}{2}\begin{pmatrix}1-i\\1+i\end{pmatrix}$, and $|v_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

i. Write $\langle v_1|$ and $\langle v_2|$ in vector notation.
ii. Show that both $|v_1\rangle$ and $|v_2\rangle$ are normalised, i.e. $\sqrt{\langle v_1|v_1\rangle} = \sqrt{\langle v_2|v_2\rangle} = 1$.
iii. Calculate the inner products $\langle v_1|v_2\rangle$ and $\langle v_3|v_1\rangle$. Are $|v_1\rangle$ and $|v_2\rangle$ orthogonal?
iv. Show that the set $\{|v_1\rangle, |v_2\rangle\}$ satisfies all the conditions of an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$.
v. Write $|v_3\rangle$ as a linear combination of $|v_1\rangle$ and $|v_2\rangle$.

(b) A general state can be represented by the superposition

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle,$$

where $\theta \in [0,\pi]$, $\varphi \in [0, 2\pi)$, and $\{|0\rangle, |1\rangle\}$ is the computational basis.

i. Prove that $|\psi\rangle$ is normalised.
ii. Find the values of θ and φ such that
   A. $|\psi\rangle = |v_3\rangle$,
   B. $|\psi\rangle = e^{-i\pi/4}|v_1\rangle$.

Problem 2: Quantum operations

Some important linear operators in quantum computing are the three *Pauli* operators

$$X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad Y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

(a) Prove the following properties of the Pauli operators.

i. They are self-adjoint, i.e. $X = X^\dagger$, $Y = Y^\dagger$, and $Z = Z^\dagger$.
ii. They are self-inverse, i.e. $X^2 = I$, $Y^2 = I$, and $Z^2 = I$, where I is the identity operator.
iii. The operators Y and Z anticommute, i.e. $YZ = -ZY$.

(b) Consider a linear operator defined by

$$U \equiv \frac{Y+Z}{\sqrt{2}}.$$

Using properties from the previous part, show that U is unitary, i.e. $U^\dagger U = UU^\dagger = I$.

(c) Calculate the action of the operator U on the vectors

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \quad |{+i}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \quad |{-i}\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix}.$$

Problem 3: Tensor product

(a) Consider the quantum state

$$|\psi\rangle = \frac{\sqrt{5}}{5}|0\rangle + \frac{2\sqrt{5}}{5}|1\rangle .$$

i. Express $|\psi\rangle^{\otimes 2}$ in Dirac notation, where $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$.

ii. Express $|+\rangle|+\rangle|-\rangle$ in Dirac notation, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

(b) Consider the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

i. Express the tensor products $X \otimes I$ and $I \otimes X$ as two 4×4 matrices.

ii. Express the tensor products $X \otimes Z$ and $Z \otimes Y$ as matrices, then calculate the matrix multiplication $(X \otimes Z)(Z \otimes Y)$.

iii. Calculate the matrices XZ and ZY, and hence verify the special case

$$(X \otimes Z)(Z \otimes Y) = (XZ) \otimes (ZY)$$

of the more general identity $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

(c) Prove that if A and B are projection operators then $A \otimes B$ is a projection operator.

(d) Prove that if A and B are unitary operators then $A \otimes B$ is a unitary operator. You may use the property that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Problem 4: Quantum measurements

(a) For each the following states, calculate the probabilities p_0 and p_1 of obtaining outcomes 0 and 1 from a measurement in the computational basis $\{|0\rangle, |1\rangle\}$, and the probabilities p_+ and p_- of obtaining outcomes + and − from a measurement in the basis $\{|+\rangle, |-\rangle\}$.

i. $|\psi_1\rangle = |1\rangle$.

ii. $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

iii. $|\psi_3\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha \neq 0$.

(b) If the outcome of a measurement in the computational basis was 0, which of $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ were possible states of the system immediately before the measurement took place?

Problem 5: Mixed states

(a) Consider the pure state formed by equal superposition of $|0\rangle$ and $|1\rangle$,

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and the *maximally mixed* state whose density matrix is given by

$$\sigma = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}.$$

i. Show that the density matrix $\rho = |\psi\rangle\langle\psi|$ of the pure state $|\psi\rangle$ is given by

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).$$

ii. For the two mixed states ρ and σ, calculate the probabilities of obtaining outcomes 0 and 1 from a measurement in the computational basis $\{|0\rangle, |1\rangle\}$.

iii. For the two mixed states ρ and σ, calculate the probabilities of obtaining outcomes + and − from a measurement in the basis $\{|+\rangle, |-\rangle\}$.

iv. Comment on the distinguishability of the two states $|\psi\rangle$ and σ with respect to the two measurement bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$.

(b) Recall that the density matrix ρ for a statistical ensemble $\{(p_1, |\psi_1\rangle), \ldots, (p_n, |\psi_n\rangle)\}$ in which each pure state $|\psi_j\rangle$ occurs with probability p_j is defined by

$$\rho = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j|.$$

i. Calculate the density matrix for the ensemble $\left\{\left(\frac{2}{3}, |0\rangle\right), \left(\frac{1}{3}, |1\rangle\right)\right\}$.

ii. Calculate the density matrix for the ensemble $\left\{\left(\frac{1}{3}, |0\rangle\right), \left(\frac{1}{3}, |+\rangle\right), \left(\frac{1}{3}, |-\rangle\right)\right\}$.

iii. Does there exist a measurement allowing an experimenter to distinguish between these two ensembles? Justify your answer.