Problem 1

Consider the four Bell states

\[ |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \]
\[ |\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \]

Those maximally entangled states form an orthonormal basis of the two-qubit Hilbert space \( \mathcal{H} = \mathbb{C}^4 \).

(a) Verify that the Bell states form an orthonormal family of states, i.e., that they are pair-wise orthogonal, and each of them is normalised.

(b) Simplify the following:

i. \( X \otimes X |\Psi^\rangle \)

ii. \( X \otimes Z |\Psi^\rangle \)

iii. \( Z \otimes X |\Psi^\rangle \)

iv. \( Z \otimes Z |\Psi^\rangle \)

Problem 2

Consider the CHSH "game" described in the lecture. Assume that Alice and Bob share the quantum state

\[ |\Phi^\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \]

Recall that

\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

If \( x = 0 \) Alice measures the observable \( A_0 = Z \), and if \( x = 1 \) Alice measures the observable \( A_1 = X \). If \( y = 0 \) Bob measures the observable \( B_0 = \frac{1}{\sqrt{2}}(X + Z) \), and if \( y = 1 \) Bob measures the observable \( B_1 = \frac{1}{\sqrt{2}}(X - Z) \).

(a) Compute the correlator \( E_{00} = \langle \Phi^+ | A_0 \otimes B_0 |\Phi^\rangle = \langle \Phi^+ | Z \otimes \frac{1}{\sqrt{2}}(X + Z) |\Phi^\rangle \).

(b) Compute the quantity \( \beta = E_{00} - E_{01} + E_{10} + E_{11} \) and show that it attains the maximum Bell inequality violation \( 2\sqrt{2} \), as given in the lectures.

Problem 3

Consider the same setting of the game as in Problem 1, but with the difference that now Alice and Bob share the state

\[ |\psi\rangle = \frac{1}{\sqrt{3}} |00\rangle + \sqrt{\frac{2}{3}} |11\rangle. \]

Compute the quantity \( \beta \) in this case.
Problem 4

Consider the same setting of the game as in Problem 1, but now Alice and Bob share a mixed state \( \rho \) that is given by the ensemble where with probability \( p_1 = 1/4 \) the state is

\[
|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},
\]

with probability \( p_2 = 1/4 \) the state is

\[
|\psi_2\rangle = \frac{1}{\sqrt{3}} |00\rangle + \frac{\sqrt{2}}{3} |11\rangle,
\]

and with probability \( p_3 = 1/2 \) the state is

\[
|\psi_3\rangle = |0\rangle \otimes |+\rangle.
\]

(a) Compute the correlator \( E_{01}(\rho) = \text{tr}[\rho (Z \otimes X - Z \sqrt{2})] \).

(b) Determine the quantity \( \beta \) corresponding to this realisation of the CHSH game.

Problem 5

In this problem, we will derive the Schmidt decomposition for a two-qubit bipartite system. That is, for any two-qubit bipartite state \( |\psi\rangle_{AB} \), there exist orthonormal bases \( \{|e_1\rangle, |e_2\rangle\} \) and \( \{|f_1\rangle, |f_2\rangle\} \) for the single-qubit systems \( A \) and \( B \) respectively and positive constants \( c_1 \) and \( c_2 \) such that

\[
|\psi\rangle_{AB} = c_1 |e_1\rangle \otimes |f_1\rangle + c_2 |e_2\rangle \otimes |f_2\rangle.
\]

(a) Consider any two orthonormal bases for the systems \( A \) and \( B \), \( \{|a_1\rangle, |a_2\rangle\} \) and \( \{|b_1\rangle, |b_2\rangle\} \) respectively. Write \( |\psi\rangle_{AB} \) in the matrix representation \( M_\psi \) with respect to these bases.

(b) Consider the singular value decomposition \( M_\psi = U\Sigma V^\dagger \), where \( U \) and \( V \) are \( 2 \times 2 \) unitaries and \( \Sigma \) is a diagonal matrix whose entries are the joint eigenvalues of the bipartite system. From this decomposition deduce \( \{|e_1\rangle, |e_2\rangle\} \) and \( \{|f_1\rangle, |f_2\rangle\} \). What are the constants \( c_1 \) and \( c_2 \)?

Problem 6

For a general quantum state \( |\psi\rangle \), the number of nonzero constants (Schmidt coefficients) \( c_i \) in its Schmidt decomposition is called the “Schmidt number” for the state \( |\psi\rangle \).

(a) Prove that a pure state \( |\psi\rangle_{AB} \) of a two-qubit bipartite system is entangled if and only if its Schmidt number is greater than 1.

(b) Suppose that \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are two states of a two-qubit bipartite system (with components \( A \) and \( B \)) having identical Schmidt coefficients. Show that there are unitary transformations \( U \) on system \( A \) and \( V \) on system \( B \) such that \( |\psi_1\rangle = (U \otimes V) |\psi_2\rangle \).