Quantum Cyber Security
Lecture 11: Secure Two-Parties Functionalities

Petros Wallden

University of Edinburgh

4th March 2025

This Lecture

1. What is Secure Multiparty Computation
2. Basic Primitives and Their Relations
3. Information Theoretic Security: Classical Impossibility
4. Could Quantum Communications achieve ITS: a naive attempt
5. Information Theoretic Security: Quantum Impossibility
6. Side-Stepping the No-Go Results

The Millionaire's Problem

The Problem

Two millionaires (Alice and Bob) want to:

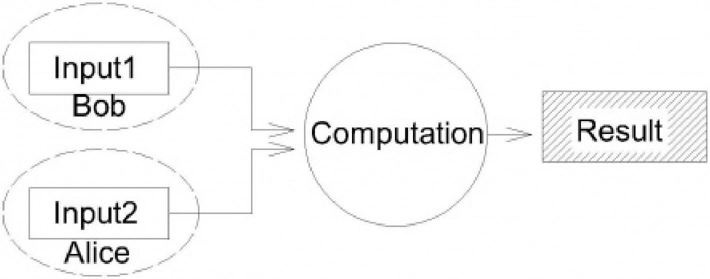
1. Determine **who is wealthier** ($a \stackrel{?}{\geq} b$)
2. Not reveal anything else about their properties

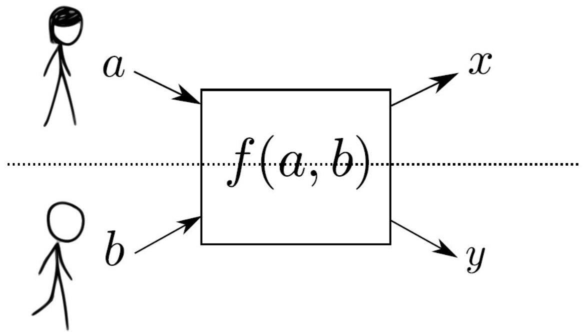
The Millionaire's Problem

The Problem

Two millionaires (Alice and Bob) want to:

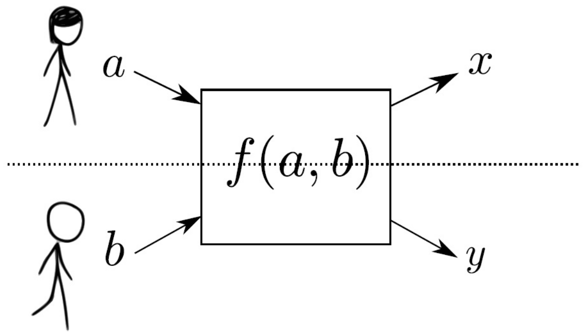
1. Determine **who is wealthier** ($a \overset{?}{\geq} b$)
2. Not reveal anything else about their properties





Some figures taken from F. Dupuis

$$f(a,b) = (x,y)$$

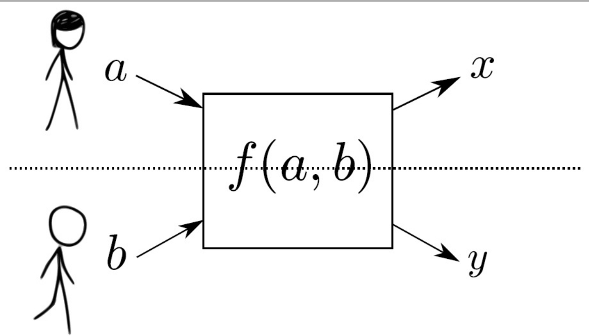


Some figures taken from F. Dupuis

$$f(a,b) = (x,y)$$

Example: Function $f(a,b) = (a \wedge b, a \wedge b)$

Secure Multiparty Computation

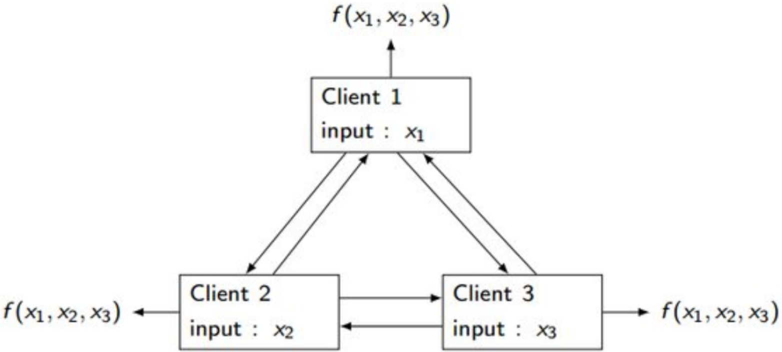


Some figures taken from F. Dupuis

$$f(a,b) = (x,y)$$

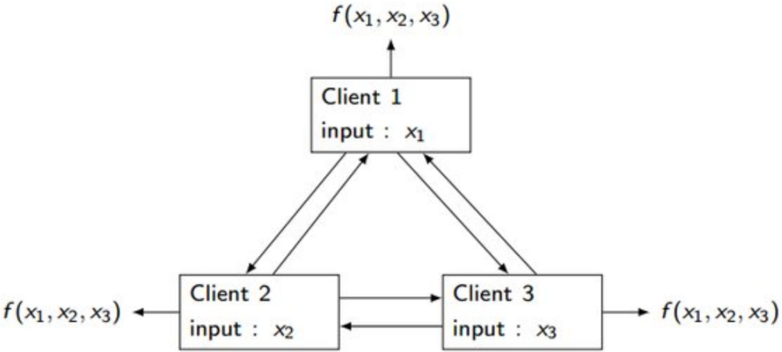
Example: Function $f(a,b) = (a \wedge b, a \wedge b)$

- If $a = 0$ Alice learns nothing on Bob's input
- If $a = 1$ Alice learns exactly Bob's input
- Protocol is **secure** because this information Alice would learn even in the ideal case!



$$f(x_1, x_2, x_3) = (y_1, y_2, y_3)$$

(In many cases the output is the same for all parties)

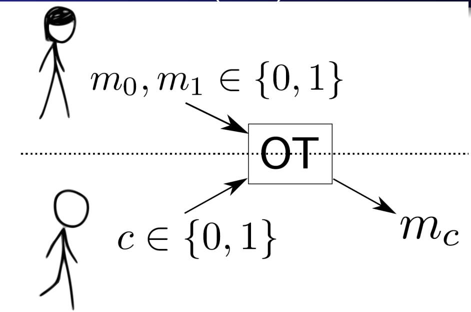


$$f(x_1, x_2, x_3) = (y_1, y_2, y_3)$$

(In many cases the output is the same for all parties)

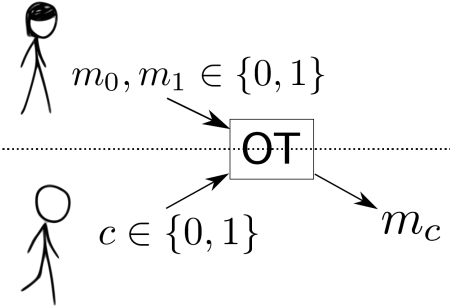
- Applications: E-voting, auctions, private information retrieval, privacy-preserving data mining, etc

1 out of 2 Oblivious Transfer (OT)



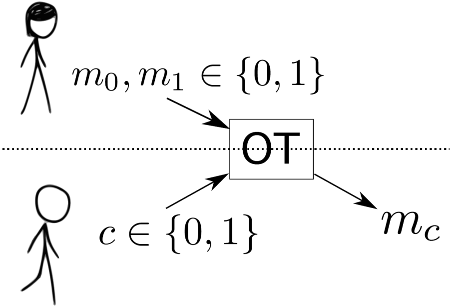
- Alice: **Inputs** two (single-bit) messages m_0, m_1
- Bob: **Inputs** a single bit c

1 out of 2 Oblivious Transfer (OT)



- Bob: Receives the message m_c (**Output**)

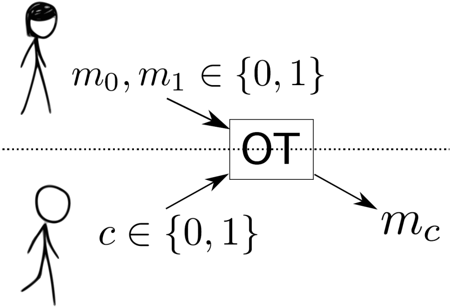
1 out of 2 Oblivious Transfer (OT)



Security

- Alice: Does **not** learn c; ie which message Bob received
- Bob: Learns **nothing** about the message $m_{c\oplus 1}$

1 out of 2 Oblivious Transfer (OT)

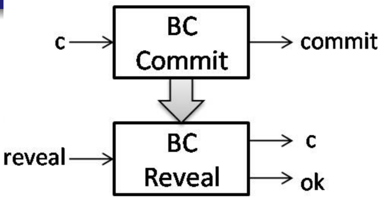


Security

- Alice: Does **not** learn c; ie which message Bob received
- Bob: Learns **nothing** about the message $m_{c \oplus 1}$

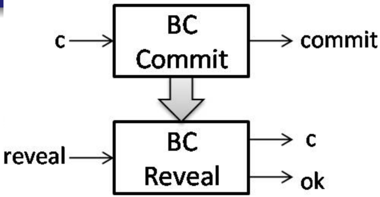
OT is Universal for Secure Multiparty Computation

Bit Commitment



Commit Phase

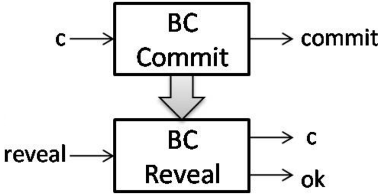
- Alice: **Inputs** a single-bit c (commits)
- Bob: receives `commit`



Reveal Phase

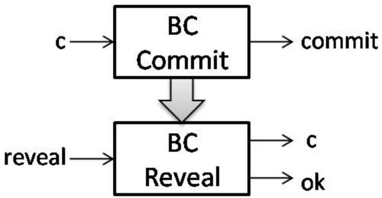
- Alice: sends the message/request "`reveal`"
- Bob: Receives c & confirmation that matches commitment

Bit Commitment



Security

- Alice: Cannot open the commitment to another value than the one she inputs in the commit phase (**Binding**)
- Bob: Learns nothing about c before `reveal` (**Concealing**)



Security

- Alice: Cannot open the commitment to another value than the one she inputs in the commit phase (**Binding**)
- Bob: Learns nothing about c before `reveal` (**Concealing**)

Implication

- BC can be constructed from OT.
- Any impossibility of BC **implies** impossibility of OT

ITS: Classical Impossibility of BC

BC Impossibility in ITS setting

It is impossible to achieve Bit-Commitment classically, with information-theoretic security (ITS)

ITS: Classical Impossibility of BC

BC Impossibility in ITS setting

It is impossible to achieve Bit-Commitment classically, with information-theoretic security (ITS)

Proof

At the end of commit phase: Bob has classical info that either:

1. Any possible `reveal` that does not abort, opens to a unique message c

ITS: Classical Impossibility of BC

BC Impossibility in ITS setting

It is impossible to achieve Bit-Commitment classically, with information-theoretic security (ITS)

Proof

At the end of commit phase: Bob has classical info that either:

1. Any possible `reveal` that does not abort, opens to a unique message c

→ Bob can brute-force trying all `reveal` and find c:
Not Concealing

ITS: Classical Impossibility of BC

BC Impossibility in ITS setting

It is impossible to achieve Bit-Commitment classically, with information-theoretic security (ITS)

Proof

At the end of commit phase: Bob has classical info that either:

1. Any possible `reveal` that does not abort, opens to a unique message c

→ Bob can brute-force trying all `reveal` and find c: **Not Concealing**

2. There exist at least two ways to open $\texttt{reveal}_c, \texttt{reveal}_{c\oplus 1}$ that opens to different message

ITS: Classical Impossibility of BC

BC Impossibility in ITS setting

It is impossible to achieve Bit-Commitment classically, with information-theoretic security (ITS)

Proof

At the end of commit phase: Bob has classical info that either:

1. Any possible `reveal` that does not abort, opens to a unique message c

→ Bob can brute-force trying all `reveal` and find c: **Not Concealing**

2. There exist at least two ways to open $\texttt{reveal}_c, \texttt{reveal}_{c\oplus 1}$ that opens to different message

→ Alice can brute-force and find both $\texttt{reveal}_c, \texttt{reveal}_{c\oplus 1}$, and thus can open commitment to either message: **Not Binding**

A Wrong Protocol for Quantum BC

Commit Phase

- Alice, to commit to 0, selects rand a state from $\{|h\rangle, |v\rangle\}$
- Alice, to commit to 1, selects rand a state from $\{|+\rangle, |-\rangle\}$
- **Alice sends Qubit to Bob that stores it**

A Wrong Protocol for Quantum BC

Reveal Phase

- Alice announces the **bit** and the **exact state** she send
- Bob measures in that basis and confirms the commitment

A Wrong Protocol for Quantum BC

Security

- Protocol is **Concealing**.
- Bob's state at the end of commit phase:

$$\rho_B = \frac{1}{2}\left(|h\rangle\langle h| + |v\rangle\langle v|\right) = \frac{1}{2}\left(|+\rangle\langle +| + |-\rangle\langle -|\right) = \frac{1}{2}\mathbb{I}$$

A Wrong Protocol for Quantum BC

Security

- Protocol is **not binding**

- If Alice follows protocol cannot de-commit to different value without being detected with some probability.

- If Alice deviates (commit phase), can postpone commitment until reveal phase. 0 **prob being detected** (see later)!

ITS: Quantum Impossibility of BC

Quantum Bit Commitment is Impossible ITS (Lo-Chau & Mayers)

It is impossible (quantumly) to achieve Bit Commitment that is Information Theoretically both **Binding** and **Concealing**

Proof

Fact (proof later): Let $|\psi\rangle_{AB}, |\chi\rangle_{AB}$ and assume that $\mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \mathrm{Tr}_A(|\chi\rangle\langle\chi|)$. There exists U_A s.t. $(U_A \otimes \mathbb{I})|\psi\rangle_{AB} = |\chi\rangle_{AB}$.

ITS: Quantum Impossibility of BC

Quantum Bit Commitment is Impossible ITS (Lo-Chau & Mayers)

It is impossible (quantumly) to achieve Bit Commitment that is Information Theoretically both **Binding** and **Concealing**

- Assume the global (Alice-Bob) state after committing to be: $0 \rightarrow |\phi_0\rangle_{AB}$; $1 \rightarrow |\phi_1\rangle_{AB}$
- Assume **perfectly concealing**: $\rho_B(0) = \mathrm{Tr}_A(|\phi_0\rangle\langle\phi_0|) = \rho_B(1) = \mathrm{Tr}_A(|\phi_1\rangle\langle\phi_1|)$

ITS: Quantum Impossibility of BC

Quantum Bit Commitment is Impossible ITS (Lo-Chau & Mayers)

It is impossible (quantumly) to achieve Bit Commitment that is Information Theoretically both **Binding** and **Concealing**

- Assume the global (Alice-Bob) state after committing to be: $0 \rightarrow |\phi_0\rangle_{AB}$; $1 \rightarrow |\phi_1\rangle_{AB}$
- Assume **perfectly concealing**: $\rho_B(0) = \mathrm{Tr}_A(|\phi_0\rangle\langle\phi_0|) = \rho_B(1) = \mathrm{Tr}_A(|\phi_1\rangle\langle\phi_1|)$
- There exist unitary $(U_A \otimes \mathbb{I})|\phi_0\rangle_{AB} = |\phi_1\rangle_{AB}$
- Alice can "commit" to 0, and then if she changes her mind can apply U_A on her qubit to commit to 1.

Not Binding at all!

ITS: Quantum Impossibility of BC

Quantum Bit Commitment is Impossible ITS (Lo-Chau & Mayers)

It is impossible (quantumly) to achieve Bit Commitment that is Information Theoretically both **Binding** and **Concealing**

Fact (proof later): Let $|\psi\rangle_{AB}, |\chi\rangle_{AB}$ and assume that $\mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \mathrm{Tr}_A(|\chi\rangle\langle\chi|)$. There exists U_A s.t. $(U_A \otimes \mathbb{I})|\psi\rangle_{AB} = |\chi\rangle_{AB}$.

- Schmidt Decomposition: $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\,|e_i\rangle_A \otimes |f_i\rangle_B$ where $|e_i\rangle_A, |f_i\rangle_B$ eigenvectors of reduced matr. $\mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|_{AB})$; $\mathrm{Tr}_A(|\psi\rangle_{AB}\langle\psi|_{AB})$ resp, and λ_i joint eigenvalues.
- Having same reduced (B) states means that the second eigenvectors (and eigenvalues) of ψ, χ are the same
- U_A is simply mapping the one local basis to the other: $U_A|e_i^{\psi}\rangle = |e_i^{\chi}\rangle$ (always possible)

Quantum Bit Commitment is Impossible ITS (Lo-Chau & Mayers)

It is impossible (quantumly) to achieve Bit Commitment that is Information Theoretically both **Binding** and **Concealing**

Approximate Concealing:

- Let $\rho_B(0) \stackrel{\epsilon}{\approx} \rho_B(1)$ in trace-distance
- Then following same argument can show that the protocol is **at most ϵ-binding**

Quantum Bit Commitment is Impossible ITS (Lo-Chau & Mayers)

It is impossible (quantumly) to achieve Bit Commitment that is Information Theoretically both **Binding** and **Concealing**

Attack on Naive Protocol:

- Alice sends one side of a Bell pair to Bob:

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|hh\rangle + |vv\rangle) = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$$

- Bob sees the same reduced matrix $\rho_B = \frac{1}{2}\mathbb{I}$
- Alice can **choose her bit later**:
  Commits to 0 Alice measures in $\{|h\rangle, |v\rangle\}$ basis
  Commits to 1 Alice measures in $\{|+\rangle, |-\rangle\}$ basis
- Alice essentially chooses to apply H or not, before measuring in computational basis
- Bob cannot distinguish this from the ideal protocol

Side-Stepping the Impossibility Results

It is **impossible to side-step** without making some **relaxation in security requested**

Note: Majority attempts **are wrong**. Check if it is clearly stated how one evades the Lo-Chau and Mayers Thm.

Side-Stepping the Impossibility Results

It is **impossible to side-step** without making some **relaxation in security requested**

Note: Majority attempts **are wrong**. Check if it is clearly stated how one evades the Lo-Chau and Mayers Thm.

- Bounded Storage Model: Assume adversary cannot store quantum information for long time (or for more than a fixed number of qubits).
- The Lo-Chau-Mayers attack (de-committing) would require to store a large system until the `reveal` phase (which can be later than the bounds of storage).

Side-Stepping the Impossibility Results

It is **impossible to side-step** without making some **relaxation in security requested**

Note: Majority attempts **are wrong**. Check if it is clearly stated how one evades the Lo-Chau and Mayers Thm.

- Relativistic: Protocol is performed by teams located in different spacetime locations. Parties cannot communicate faster-than-the-speed-of-light.
- Commitment has **to be opened within a fixed time** period (expires/stops being binding after that)
- The Lo-Chau-Mayers attack (de-committing) would involve applying a unitary on the joint system that during the protocol is **not located in a single spacetime location** (lab).