Quantum Cyber Security
Lecture 15: Post-Quantum Cryptography II

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1. Lattice Problems:

   Learning-With-Errors (LWE)

   Shortest-Vector Problem (SVP)

2. LWE-based Public-Key Encryption (Regev's)

This Lecture: Lattice-Based Crypto & Regev's Encryption

1. Lattice Problems:

   Learning-With-Errors (LWE)

   Shortest-Vector Problem (SVP)

2. LWE-based Public-Key Encryption (Regev's)

   Notation colour code: parameters and functions: public (blue), private (red), secret but not used later (brown)

The Learning-With-Errors (LWE) Problem

- Parameters:
  - Vectors in $\mathbb{Z}_q^n$
  - $n \in \mathbb{N}$ dimension of vectors
  - q is a prime number where additions are carried over $\bmod q$
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Where we have:

- Random public vectors $\vec{a}_i$
- A single random **secret** vector $\vec{s}$ that we want to find
- Small error terms e_i (sampled from a distribution that w.h.p. is small, i.e. $e \ll q$) that are kept secret
- Public scalars $b_i := \vec{a}_i \cdot \vec{s} + e_i$
- $i \in \{1, 2, \ldots, m\}$

The Learning-With-Errors (LWE) Problem: Example

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14s_1 + 15s_2 + 5s_3 + 2s_4 &\approx 8 \pmod{17} \\
13s_1 + 14s_2 + 14s_3 + 6s_4 &\approx 16 \pmod{17} \\
6s_1 + 10s_2 + 13s_3 + 1s_4 &\approx 3 \pmod{17} \\
&\vdots \\
9s_1 + 5s_2 + 9s_3 + 6s_4 &\approx 9 \pmod{17}
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where $\vec{a_1} = (14, 15, 5, 2)$ and $b_1 = 8$, etc.

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- Find the secret vector $\vec{s} = (s_1, s_2, s_3, s_4)$

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- Alternative Version:

Decisional LWE Problem

Can a (quantum) poly-time adversary distinguish between LWE samples $(\vec{a}_i, b_i)$ and random samples $(\vec{a}_i, r_i)$; $r_i \leftarrow \mathbb{Z}_q$?

The Shortest Vector Problem (SVP)

Parameters:

- n dimension vector space
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SVP Problem

Find the shortest (non-zero) integer linear combination of basis vectors: $\vec{SV} := \vec{b}_1 x_1 + \ldots + \vec{b}_k x_k$, where $(x_1, \ldots, x_k) \in Z^k \setminus \{0\}$. We define $\lambda(L) := \|\vec{SV}\|$.

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SVP$_\beta$ Problem (Approximate)

Find a non-zero integer vector with length $\beta\lambda(L)$.

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SVP_β Problem (Approximate)

Find a non-zero integer vector with length $\beta\lambda(L)$.

GapSVP_β Problem

Determine whether the shortest vector has $\lambda(L) \leq 1$ or $\lambda(L) \geq \beta$

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- Approximate versions are also **believed to be hard** (but not proven – hardness depends on the approximation β)
- Regev's encryption scheme (next) is secure provided that the decisional-LWE is hard.

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   - **Private Key:** $\vec{s} \leftarrow Z_q^n$

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1. KeyGen:
   - **Private Key:** $\vec{s} \leftarrow Z_q^n$
   - **Public Key:** m LWE samples $(\vec{a}_i, b_i)$, where:
   - $b_i = \vec{a}_i \cdot \vec{s} + e_i$
   - $\vec{a}_i \leftarrow Z_q^n \ \forall \ i$
   - e_i random small numbers (sampled from normal distribution with standard deviation αq).

LWE-based Encryption Scheme (Regev's)

2. $\mathrm{Enc}((\vec{a}_i, b_i), \mu)$:
   - For single bit message $\mu \in \{0, 1\}$
   - Choose a random subset S of indices $\{1, \ldots, m\}$ (out of the 2^m possible subsets).
   - Compute $\vec{a} := \sum_{i \in S} \vec{a}_i$ and $b := \sum_{i \in S} b_i$
   - Output pair $(\vec{a}, c)$, where $c := b + \mu \left\lfloor \frac{q}{2} \right\rfloor$

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3. $\text{Dec}((\vec{a}, c), \vec{s})$:
   - Compute $c - \vec{a} \cdot \vec{s}$
   - Check whether outcome is closer to 0 or $\frac{q}{2}$ (oper. done $\bmod q$)
   - Output $\mu = 0$ if closer to zero, and $\mu = 1$ otherwise

- **Correctness:** We consider $\mathrm{Dec}(\mathrm{Enc}((\vec{a}_i, b_i), \mu), \vec{s})$.

$$\begin{aligned} c - \vec{a} \cdot \vec{s} &= b + \mu \left\lfloor \frac{q}{2} \right\rfloor - \vec{a} \cdot \vec{s} \\ &= \sum_{i \in S} (\vec{a}_i \cdot \vec{s} + e_i) + \mu \left\lfloor \frac{q}{2} \right\rfloor - \left(\sum_{i \in S} \vec{a}_i \right) \cdot \vec{s} \\ &= \sum_{i \in S} e_i + \mu \left\lfloor \frac{q}{2} \right\rfloor \end{aligned}$$

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Provided e_i's are small enough, this is closer to 0 when $\mu = 0$ and to $\frac{q}{2}$ otherwise.

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  - Thus $c = b + \mu \left\lfloor \frac{q}{2} \right\rfloor$ looks like $r + \mu \left\lfloor \frac{q}{2} \right\rfloor$ to the adversary.

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- We define rings and give another ring lattice-based cryptosystem at the next lecture.