Quantum Cyber Security
Lecture 2: Quantum Information Basics I

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Motivation: From Bit-strings to Qubit-strings

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- Q: Why we have speed-up?
  A: Like classical probabilistic algorithms BUT with **complex** "probabilities"

A Qubit is a 2-dimensional unit vector

- For formal definitions look at: Math Supplement; Nielsen & Chuang; or first lectures of IQC (`https://opencourse.inf.ed.ac.uk/iqc/course-materials/schedule` or an older version `http://pwallden.gr/courseiqc.asp`)

A Qubit is a 2-dimensional unit vector

- We will denote a vector $\vec{v}$ as $|v\rangle$

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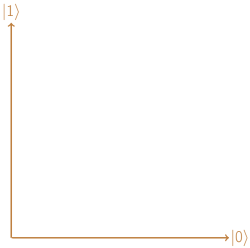
- The unit vectors in the x-axis as $|0\rangle$ and in the y-axis as $|1\rangle$



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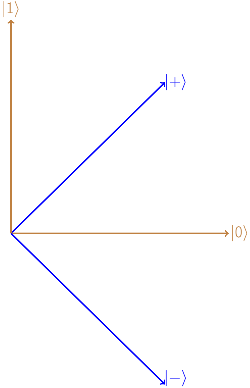
- Another basis (45% rotated) is given by the vectors $\{\left|+\right\rangle, \left|-\right\rangle\}$, where $\left|+\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle)$; $\left|-\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle)$



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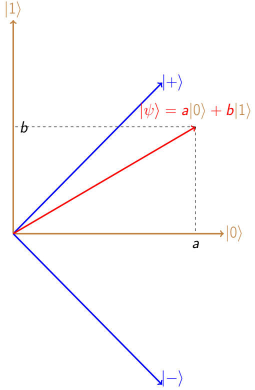
- General Qubit: $|\psi\rangle = a|0\rangle + b|1\rangle$ where $|\,|\psi\rangle\,|^2 = 1 = |a|^2 + |b|^2$ and a, b are **complex numbers** in general



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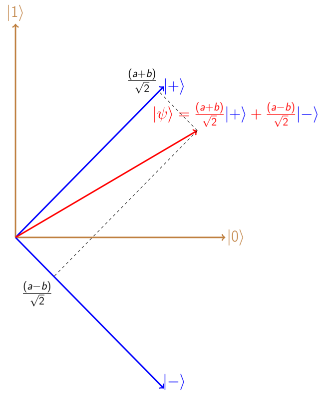
- Can be expressed in the blue basis: $|\psi\rangle = \frac{(a+b)}{\sqrt{2}}|+\rangle + \frac{(a-b)}{\sqrt{2}}|-\rangle$



Definitions with Examples

- **Vector** (notation) $|\psi\rangle$ is called "**ket**".
  Example: $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$
- **Dual vector** is denoted $\langle\psi|$ and is called "**bra**". Coefficients are complex conjugate of the coefficients of the vectors
  Example: $\langle\psi| = a^*\,\langle 0| + b^*\,\langle 1|$
- **Inner product** (c.f. dot-product) is taken between a **vector** and a **dual vector** (c.f. "**bra-ket**").

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- **Inner product** (c.f. dot-product) is taken between a **vector** and a **dual vector** (c.f. "**bra-ket**").
- Orthogonal vectors have zero inner product so:
  $\langle 0|1\rangle = \langle 1|0\rangle = 0$ and $\langle 0|0\rangle = \langle 1|1\rangle = 1$
- Example: $\langle\psi_2|\psi_1\rangle = a_2^* a_1 + b_2^* b_1 = \langle\psi_1|\psi_2\rangle^*$
  Let $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$; $|\psi_2\rangle = \frac{1}{2}(i|0\rangle + \sqrt{3}|1\rangle)$
  Check: $\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1$ and
  $\langle\psi_2|\psi_1\rangle = \frac{\sqrt{3}-i}{2\sqrt{2}}$; $\langle\psi_1|\psi_2\rangle = \frac{\sqrt{3}+i}{2\sqrt{2}}$

In matrix notation:

Vectors: $|\psi_1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and **Dual Vectors**: $\langle\psi_2| = \begin{pmatrix} a_2^* & b_2^* \end{pmatrix}$

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- Operations (gates) and Observables correspond to **linear maps**
  (Complex valued) Matrix with matrix elements m_{ij}
  $M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} = \sum_{i,j\in\{0,1\}} m_{ij} |i\rangle \langle j|$
- **Outer Product** between a vector and a dual vector (opposite order of inner "**ket-bra**"):
  $|\psi_1\rangle \langle\psi_2| = \begin{pmatrix} a_1 a_2^* & a_1 b_2^* \\ b_1 a_2^* & b_1 b_2^* \end{pmatrix}$

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Example: $A = \begin{pmatrix} 1 & 1+i \\ 2 & 3+2i \end{pmatrix} = |0\rangle\langle 0| + (1+i)|0\rangle\langle 1| +$
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  $(A|v\rangle)^\dagger = \langle v|A^\dagger$ and
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  Example: The matrix A above is NOT Hermitian, while the matrix B **is**
  $B = \begin{pmatrix} 1 & 2+3i \\ 2-3i & 5 \end{pmatrix} = B^\dagger$

Definitions with Examples

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  These operators, restrict/project a vector to some subspace of the total Hilbert space

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  Unitary operators preserve the inner product of vectors
  $\langle v|w\rangle = \langle v|U^\dagger U|w\rangle$

- **Operations**/gates/channels for (pure) quantum states are unitaries and they map quantum states to quantum states $U|\psi\rangle = |\phi\rangle$ noting that $\langle\phi|\phi\rangle = 1 = \langle\psi| U^\dagger U |\psi\rangle = \langle\psi|\psi\rangle$

  Examples: Identity I; Pauli X, Y and Z gates

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hadamard H

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

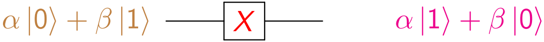
Example:

- The quantum NOT-gate is the Pauli X:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Acts as the NOT-gate to computational basis vectors: $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$

For a general qubit: $\alpha\,|0\rangle + \beta\,|1\rangle \rightarrow \alpha\,|1\rangle + \beta\,|0\rangle$



- **Measurement** (projective) for pure states
- Computational basis: Given the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ we measure in the $\{|0\rangle, |1\rangle\}$ basis

- With probability $|\alpha|^2$ we get the outcome 0; output state is $|0\rangle$
- With probability $|\beta|^2$ we get the outcome 1; output state is $|1\rangle$

- General basis: We express the state in that basis and repeat

  Example: To measure in the $\{|+\rangle, |-\rangle\}$ basis we re-express $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in that basis:
  $|\psi\rangle = \frac{(a+b)}{\sqrt{2}}|+\rangle + \frac{(a-b)}{\sqrt{2}}|-\rangle$

- Outcome $+$ with prob $|\frac{(a+b)}{\sqrt{2}}|^2$ and final state $|+\rangle$
- Outcome $-$ with prob $|\frac{(a-b)}{\sqrt{2}}|^2$ and final state $|-\rangle$

- **Check:** What happens if one measures $|+\rangle$ in the $\{|0\rangle, |1\rangle\}$ and in the $\{|+\rangle, |-\rangle\}$ bases?

- Measurement formally: Given two projection P_1, P_2 where $P_1 + P_2 = I$

  - Outcome cor. to P_1 with probability $\langle\psi| P_1 |\psi\rangle$ and output state $(P_1 |\psi\rangle) \frac{1}{\sqrt{\langle\psi|P_1|\psi\rangle}}$

  - Outcome cor. to P_2 with probability $\langle\psi| P_2 |\psi\rangle$ and output state $(P_2 |\psi\rangle) \frac{1}{\sqrt{\langle\psi|P_2|\psi\rangle}}$

- Note: the sum of probabilities is one:

$$\langle\psi| P_1 |\psi\rangle + \langle\psi| P_2 |\psi\rangle = \langle\psi| (P_1 |\psi\rangle + P_2 |\psi\rangle) =$$

$$= \langle\psi| (P_1 + P_2) |\psi\rangle = \langle\psi| I |\psi\rangle = 1$$

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  The trace of an operator is invariant under unitary similarity transformations $A \to UAU^\dagger$
  $\mathrm{Tr}(UAU^\dagger) = \mathrm{Tr}(U^\dagger UA) = \mathrm{Tr}(A)$

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  The state of this system is described by the following density matrix: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

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    1. **Classical randomness** since we do not know which is the (real) pure quantum state. This randomness is due to the lack of knowledge that we (the observers) have. Is the same with the randomness of classical physics (epistemic).

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  2. Fundamental **quantum randomness**. This is due to the fact that even if we know the exact pure quantum state (have maximum information about the system), multiple outcomes may occur.

- Mixed state:

$$\rho_1 = 1/2\,|0\rangle\langle 0| + 1/2\,|1\rangle\langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Example: Classical Vs Quantum Randomness

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- Measured in computational basis $\{\left|0\right\rangle, \left|1\right\rangle\}$ both give same probabilities (but for ρ_1 is classical randomness while for ρ_2 is quantum randomness).
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- Difference between maximally mixed and equal superposition!

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3. has unit trace $\mathrm{Tr}(\rho) = 1$

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Exercise: Check that these conditions are satisfied

1. for pure density matrices
2. for density matrices of the form $\rho = \sum_i p_i \left|\psi\right\rangle \left\langle\psi\right|$

Mixed States

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Example: $\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$

Ensemble 1: $\{p(0) = 3/4, |0\rangle, p(1) = 1/4, |1\rangle\}$

Ensemble 2: $\{p(a) = 1/2, |a\rangle, p(b) = 1/2, |b\rangle\}$ where

$|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$

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Check that: $\rho = \frac{1}{2}\,|a\rangle\langle a| + \frac{1}{2}\,|b\rangle\langle b| = \frac{3}{4}\,|0\rangle\langle 0| + \frac{1}{4}\,|1\rangle\langle 1|$

Operations and Measurements for Mixed States

- More information will be given in later lectures.
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  Example: Evolve by X the state $\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$.

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- **Measurements:** Projective measurement P_1, P_2, at state ρ.
- Probability of outcomes $p_1 = \mathrm{Tr}(P_1\rho)$; $p_2 = \mathrm{Tr}(P_2\rho)$
- State after measurement

$$\rho_1 = P_1\rho P_1 \frac{1}{\mathrm{Tr}(P_1\rho)} \; ; \; \rho_2 = P_2\rho P_2 \frac{1}{\mathrm{Tr}(P_2\rho)}$$

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- Possible values of measuring the observable are the **eigenvalues**
- Probability of each outcome is given by **projecting on the corresponding eigenspace**