

Quantum Cyber Security

Lecture 8: Quantum Key Distribution II

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- From **QBER** to **secure key distribution** (general expression and how to use it)
- **Simplifying assumptions** (physical restrictions, classical efficiency, adversary's limitations, composability)
- **Security proof** for the basic BB84 protocol
- **Classical post-processing** and its cost

- **General Expression:**

$$R = \frac{Q}{2} (\xi H(A : B) - S(A : E) - \Delta(n, \epsilon))$$

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- R is the secret **key-rate**: Expected secret bits per qubit sent.
- Q is the prob that sent single-photons are detected (not lost)
- factor $\frac{1}{2}$ is due to the raw key that includes only the positions that Alice and Bob measured in same basis
- ξ is due to non-ideal classical post-processing (IR and PA)
- $\Delta(n, \epsilon)$ is a factor due to finite-size effects (measured value differing from expectation)

From QBER to Secure Key Distribution

- For simplicity we consider: **perfect detection**, **ideal post-processing** and **asymptotic limit**

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- Using details of BB84 protocol we get:

$$R_{\text{BB84}} = \frac{1}{2} (1 - h(e_b) - h(e_p)) \quad (1)$$

where e_b and e_p are the average errors in the $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ bases and $h(p) := -p \log_2 p - (1-p) \log_2 (1-p)$ is the binary entropy

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- If the errors in different bases equal and equal to the QBER ($e_b = e_p = D$) we finally get:

$$R_{\text{BB84}} = \frac{1}{2} (1 - 2h(D)) \quad (2)$$

Examples: How to compute key rate (ideal case)

- **Example 1:** Given $e_b = 0.05$, $e_p = 0.1$ find the rate.

$$R_{\text{BB84}} = \frac{1}{2} (1 - h(0.05) - h(0.1)) = \frac{1}{2} (1 - 0.29 - 0.47) = 0.12$$

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- **Example 3:** Does intercept, measure Z & resend attack abort?

$$e_b = 0 ; e_p = 0.5 ; R_{\text{BB84}} = \frac{1}{2} (1 - h(0.5)) = 0$$

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 - **True single-photon source:** In practise sources frequently produce pairs of (identical) photons instead of single photons (this affects the security)
 - **Fully trusted quantum devices:** Assumptions on how the preparation and measuring devices behave and what information on their workings could leak (e.g. due to a hacking/side-channel attack)

- **Finite-size effects:** Bounds on the mutual information are computed based on expectations values of observables.

Measured values differ from expectation values for finite size keys, but they converge (exponentially – cf Chernoff bounds) when the length of the string tends to infinity.

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- **Cost of classical post-processing:** Theoretical error-correction (IR) leaks information to make A', B' perfectly correlated, related with the conditional entropy $H(A|B)$

Practical error-correction leaks more bits of information (cf ξ -coefficient)

Assumptions: Adversary's model

- **Ability of adversary:** (from weaker to stronger)
 - **i.i.d. attacks:** Interacts with sent each qubit separately, independently and identically
Can reduce remarks regarding **strings of qubits** to the **expected effect on a single qubit**
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General action: $U_{BE} (\rho_B^x \otimes |0,0\rangle_E \langle 0,0|) = \sigma_{BE}^x$ and sending system B to Bob (wlog E is 2-qubit).
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 - **Collective attacks:** Uses **different private system** for each qubit, interacts with each qubit (non iid) and then measures conditionally on other previous actions
 - **Coherent Attacks:** Uses private system(s), **interacts with all passing qubits**, stores everything and **measures all systems at the end** (possibly in entangled basis)

- **Composability:** In modern crypto, security is proven in such a way that essential **properties proven** are directly maintained **when composed with other protocols** (e.g. used as subroutine in a larger protocol)

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Way to prove:

- Define **ideal properties** that protocol would have
- Any adversary has **bound probability of distinguishing** the **real protocol** from a simulated protocol that uses the **ideal protocol**
- In quantum case, bounding this probability reduces to bounding the **trace-distance** of the **real protocol** from an **ideal protocol**

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- **Simplifying Assumptions:**
 - Asymptotic limit ($N \rightarrow \infty$)
 - No losses ($Q = 1$)
 - trusted and ideal single-photon source and measuring devices
 - ideal classical post-processing ($\xi = 1$)

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 - Asymptotic limit ($N \rightarrow \infty$)
 - No losses ($Q = 1$)
 - trusted and ideal single-photon source and measuring devices
 - ideal classical post-processing ($\xi = 1$)
- **Adversarial Model:** **i.i.d.** and **non-composable**
- Proof **can be generalised** for stronger adversaries and without the simplifying assumptions, adjusting parameters and with simple protocol modifications

BB84: A basic security proof

- i.i.d. case see effects on **single qubit** (rather than strings)
- Need to bound (subject to average errors e_b, e_p):

$$R = \frac{1}{2}(H(A : B) - S(A : E))$$

- See also alternative proof later

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- Need to bound (subject to average errors e_b, e_p):

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- See also alternative proof later
- $H(A : B) = H(A) - H(A|B) = 1 - \frac{1}{2}(h(e_b) + h(e_p))$
 $H(A) = 1$ since A is chosen randomly
 $H(A) = -1/2 \log_2 \frac{1}{2} - 1/2 \log_2 \frac{1}{2} = 1$
 $H(A|B)$ when state is sent in the Z basis is
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 $H(A|B)$ when state is sent in the X basis is
 $H(A|B) = -(1 - e_p) \log_2(1 - e_p) - e_p \log_2 e_p = h(e_p)$ and happens in the other cases
- Overall: $H(A|B) = \frac{1}{2}(h(e_b) + h(e_p))$

- Need to bound $S(A : E)$. Eve has the quantum state:

$$\sigma_E^x = \text{Tr}_B (U_{BE}(\rho_B^x \otimes |0,0\rangle_E \langle 0,0|))$$

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Accessible Information: Given ensemble $F := \{(p(x), \sigma^x)\}$, the (generalised) measurement $\{M\}$, and the random variable corresponding to the measurement's outcome Y_M :

$$I_{acc}(F) = \max_M H(X : Y_M)$$

- **Holevo bound:** Given ensemble $F := \{(p(x), \sigma^x)\}$, the accessible information is upper bounded by the Holevo quantity $\chi(F)$

$$I_{acc}(F) \leq \chi(F) := S\left(\sum_x p(x)\sigma^x\right) - \sum_x p(x)S(\sigma^x)$$

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- In our case Eve's register is $d = 2^2$ and thus:

$$I_{acc} \leq \chi(F) \leq 2$$

The **maximum classical information extractable from a single qubit** (irrespective of the number of classical states encoded) is **one bit!**

- Let $\sigma_{BE} = |\psi\rangle_{BE}\langle\psi|$ be a **pure state** (global), then:

$$S(\sigma_B) := S(\text{Tr}_E(\sigma_{BE})) = S(\sigma_E) := S(\text{Tr}_B(\sigma_{BE}))$$

(See **Schmidt decomposition** for proof)

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- In our case ($F = \{p(x), \sigma_E^x\}$) for the individual terms, σ_{BE}^x is pure so we can use the entropy of the B system, which for given x is given by the resp error:

$$S(A : E) \leq I_{acc}(F) \leq \chi(F) = S(\sigma_E) - \frac{1}{2}(h(e_b) + h(e_p))$$

Leading to

$$R \geq \frac{1}{2} (H(A : B) - \chi(F)) = \frac{1}{2} (1 - S(\sigma_E))$$

- It can also be shown that $S(\sum_x \frac{1}{4} \sigma_E^x) \leq h(e_b) + h(e_p)$
- Algebraically has 2 maximum value (if Bob's state is random and independent of the state of Alice)
- This leads to the final expression given by Eq. (1)

$$R_{BB84} \geq \frac{1}{2} (1 - h(e_b) - h(e_p))$$

- This becomes negative if e_p, e_b increase (has max value -1 when these become $1/2$)
- The overall $1/2$ factor in Eq. (1) can be removed if the states sent are mainly in one (preferred) basis. This is possible if there are sufficient states sent in the other basis to have good enough statistics (cf finite-size effects)

BB84: Another proof based on entropic uncertainty

- Consider tripartite system ABE . System A is either measured in Z or X basis to result to classical variable A^Z, A^X resp
- For simplicity systems A, B are assumed to be single qubits, then the following **inequality** holds for all global states ρ_{ABE}

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- We have

$$H(A : B) - S(A : E) = S(A|E) - H(A|B) = S(A|E) - S(A|B)$$

which we can break to two terms depending the basis used:

$$\frac{1}{2} \left(S(A^Z|E) - S(A^Z|B^Z) + S(A^X|E) - S(A^X|B^X) \right)$$

- From Eq. (3) we get:

$$I(A : B) - S(A : E) \geq \frac{1}{2} \left((1 - S(A^X | B^X)) - S(A^Z | B^Z) + \right. \\ \left. + (1 - S(A^Z | B^Z)) - S(A^X | B^X) \right)$$

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- Noting that $S(A^Z | B^Z) = h(e_b)$; $S(A^X | B^X) = h(e_p)$

$$I(A : B) - S(A : E) \geq 1 - h(e_b) - h(e_p)$$

which then leads to the known expression Eq. (1)

- Once raw-key is obtained and QBER computed and threshold is achieved, we still need to classically process the resulted keys to ensure that they are **identical** between Alice and Bob and **completely secret** from Eve.

Classical Post-Processing

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The number of bits required is estimated from the mutual information $H(A : B)$ using the QBER. This **amount of information** is also **leaked to Eve**

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- **Privacy Amplification (PA)**: Use family of **universal hash functions** to ensure that the final (smaller) key Alice and Bob share, is completely secret from Eve (i.e. amplify the privacy). Map strings to smaller strings s.t. entropy $H(A''|E'')$ of new strings $A'' = g(A')$; $E'' = g(E')$ is maximum

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Due to non-ideal procedure, to ensure identical output **leaked bits are increased by a factor ξ** compared to ideal Shannon limit

- **Leftover hash lemma**: if a secret bit-string A of length n has t bits leaked (at unknown positions), then you can produce a bit string of $m \leq n - t - 2 \log_2(1/\epsilon)$ bits that is totally secret (almost optimal)

Privacy Amplification

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- **2-Universal hash family**: Let a family of functions $g_i \in G$ with $i \in S$ (cardinality of family $|S|$), where $g_i : \{U \rightarrow [m] = \{0, 1\}^m\}$:
 - 1 for fixed $A \in U$ if g_i is randomly chosen from the family, the $g_i(A)$ is uniformly distributed in $[m]$
 - 2 for any pair $A, E \in U$, if i is chosen randomly, $g_i(A), g_i(E)$ are independent variables

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- Consider a string A with $(n - t)$ -bits of randomness. If $m \leq (n - t)$ then using the 2-universal hash family G :

$$\delta[(g_i(A), i), (R, i)] \leq \epsilon$$

R uniformly random m -bit string, δ stat distance