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Tutorial 2

Problem 1: Bit-flip channel

The bit-flip channel does nothing with probability p, and flips a ket $|0\rangle$ (respectively $|1\rangle$) to a ket $|1\rangle$ (respectively $|0\rangle$) with probability $1-p$. Its effect on a state ρ is given by

$$\Phi_{BF}[\rho] = p\rho + (1-p)X\rho X,$$

where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Compute the output state of the bit-flip channel when its input is a general qubit mixed state

$$\rho = \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix}.$$

(b) What is the effect of applying the bit-flip channel with $p = 1/2$ to one of the two qubits of the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$?

Hint: Doing nothing on the first qubit and applying a bit-flip channel on the second qubit has the effect of doing nothing with probability p and applying $I \otimes X$ with probability $1-p$.

Problem 2: Depolarizing channel

The depolarizing channel applies a bit-flip, phase-flip, or both, with probability $p/4$, and does nothing with probability $1 - 3p/4$. Its effect on a state ρ is given by

$$\Phi_{DE}[\rho] = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z),$$

where

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Compute the output state of the depolarizing channel when its input is a general qubit mixed state ρ.

(b) Show that $\Phi_{DE}[\rho] = (1-p)\rho + p\mathbb{1}/2$, where $\mathbb{1}$ represents the identity matrix.

Problem 3: Amplitude damping channel

The Kraus representation of the damping channel is given by

$$\Phi_{DA}[\rho] = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger,$$

where

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

(a) Check that $\sum_k E_k^\dagger E_k = \mathbb{1}$.

(b) Compute the output state of an amplitude damping channel when its input is the state

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 4: Freedom in operator-sum representation*

Consider a quantum channel $\mathcal{L}$ defined by the operator sum representation

$$\mathcal{L}[\rho] = \sum_j E_j \rho E_j^\dagger,$$

with Kraus operators given by

$$E_0 = |0\rangle\langle 0|\,, \quad E_1 = |1\rangle\langle 1|\,.$$

Consider also a quantum channel $\mathcal{F}$ defined by the operator sum representation

$$\mathcal{F}[\rho] = \sum_j F_j \rho F_j^\dagger,$$

with Kraus operators given by

$$F_0 = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{\sqrt{2}}, \quad F_1 = \frac{|0\rangle\langle 0| - |1\rangle\langle 1|}{\sqrt{2}}.$$

(a) Given that the input state is a general qubit state

$$\rho = \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix},$$

show that the output state of the channel $\mathcal{L}$ is

$$\mathcal{L}[\rho] = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix},$$

i.e., the channel $\mathcal{L}$ destroys all coherence in the initial quantum state.

(b) Compute the output state of channel $\mathcal{F}$ when its input is the same general state ρ as in the previous part.

(c) Comment on the relation between the channels $\mathcal{L}$ and $\mathcal{F}$. What can we infer about the uniqueness of sets of Kraus operators for quantum channels?

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Problem 5: Positive Operator-Valued Measurement

Consider the set of matrices

$$M_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |0\rangle\langle 0| , \quad M_2 = \frac{\sqrt{2}}{1+\sqrt{2}} |+\rangle\langle +| ,$$
$$M_3 = \frac{1-\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |0\rangle\langle 0| + \frac{1+\frac{\sqrt{2}}{2}}{1+\sqrt{2}} |1\rangle\langle 1| - \frac{\frac{\sqrt{2}}{2}}{1+\sqrt{2}} (|0\rangle\langle 1| + |1\rangle\langle 0|)$$

(a) i. Check that the completeness relation is satisfied, i.e., $\sum_j M_j = I$.

ii. It can be shown that M_1 and M_3 are positive semidefinite matrices. Show that M_2 also is, i.e., M_2 has no negative eigenvalues.

iii. Conclude on whether or not $\{M_1, M_2, M_3\}$ is a valid positive operator-valued measurement (POVM).

(b) Suppose that you were given a qubit by Alice. All you know is that she prepared it in one of two states:

$$|\Psi_1\rangle = |1\rangle , \quad |\Psi_2\rangle = |-\rangle .$$

i. Show that the probability of getting outcome 1 with measurement $\{M_1, M_2, M_3\}$ if you received state $|\Psi_1\rangle$ is 0.

ii. Show that the probability of getting outcome 2 with measurement $\{M_1, M_2, M_3\}$ if you received state $|\Psi_2\rangle$ is 0.

iii. Compute the probability of obtaining outcome 3

A. if you received $|\Psi_1\rangle$.

B. if you received $|\Psi_2\rangle$.

iv. Discuss for each measurement outcome if you can infer something about the qubit prepared by Alice.