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Tutorial 5

Problem 1

Consider the four Bell states

$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad \left|\Phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$\left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad \left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Those maximally entangled states form an orthonormal basis of the two-qubit Hilbert space $\mathcal{H} = \mathbb{C}^4$.

(a) Verify that the Bell states form an orthonormal family of states, i.e., that they are pair-wise orthogonal, and each of them is normalised.

(b) Simplify the following:

i. $X \otimes X \left|\Psi^-\right\rangle$
ii. $X \otimes Z \left|\Psi^-\right\rangle$
iii. $Z \otimes X \left|\Psi^-\right\rangle$
iv. $Z \otimes Z \left|\Psi^-\right\rangle$

Problem 2

Consider the CHSH "game" described in the lecture. Assume that Alice and Bob share the quantum state

$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Recall that

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $x = 0$ Alice measures the observable $A_0 = Z$, and if $x = 1$ Alice measures the observable $A_1 = X$. If $y = 0$ Bob measures the observable $B_0 = \frac{1}{\sqrt{2}}(X + Z)$, and if $y = 1$ Bob measures the observable $B_1 = \frac{1}{\sqrt{2}}(X - Z)$.

(a) Compute the correlator $E_{00} = \langle\Phi^+| A_0 \otimes B_0 |\Phi^+\rangle = \langle\Phi^+| Z \otimes \frac{1}{\sqrt{2}}(X + Z) |\Phi^+\rangle$.

(b) Compute the quantity $\beta = E_{00} - E_{01} + E_{10} + E_{11}$ and show that it attains the maximum Bell inequality violation $2\sqrt{2}$, as given in the lectures.

Problem 3

Consider the same setting of the game as in Problem 1, but with the difference that now Alice and Bob share the state $|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$. Compute the quantity β in this case.

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Problem 4

Consider the same setting of the game as in Problem 1, but now Alice and Bob share a mixed state ρ that is given by the ensemble where with probability $p_1 = 1/4$ the state is

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

with probability $p_2 = 1/4$ the state is

$$|\psi_2\rangle = \frac{1}{\sqrt{3}} |00\rangle + \sqrt{\frac{2}{3}} |11\rangle ,$$

and with probability $p_3 = 1/2$ the state is

$$|\psi_3\rangle = |0\rangle \otimes |+\rangle .$$

(a) Compute the correlator $E_{01}(\rho) = \mathrm{tr}[\rho(Z \otimes \frac{X-Z}{\sqrt{2}})]$.

(b) Determine the quantity β corresponding to this realisation of the CHSH game.

Problem 5

In this problem, we will derive the Schmidt decomposition for a two-qubit bipartite system. That is, for any two-qubit bipartite state $|\psi\rangle_{AB}$, there exist orthonormal bases $\{|e_1\rangle, |e_2\rangle\}$ and $\{|f_1\rangle, |f_2\rangle\}$ for the single-qubit systems A and B respectively and positive constants c_1 and c_2 such that $|\psi\rangle_{AB} = c_1 |e_1\rangle \otimes |f_1\rangle + c_2 |e_2\rangle \otimes |f_2\rangle$.

(a) Consider any two orthonormal bases for the systems A and B, $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ respectively. Write $|\psi\rangle_{AB}$ in the matrix representation M_ψ with respect to these bases.

(b) Consider the singular value decomposition $M_\psi = U\Sigma V^\dagger$, where U and V are 2×2 unitaries and Σ is a diagonal matrix whose entries are the joint eigenvalues of the bipartite system. From this decomposition deduce $\{|e_1\rangle, |e_2\rangle\}$ and $\{|f_1\rangle, |f_2\rangle\}$. What are the constants c_1 and c_2?

Problem 6

For a general quantum state $|\psi\rangle$, the number of nonzero constants (Schmidt coefficients) c_i in its Schmidt decomposition is called the "Schmidt number" for the state $|\psi\rangle$.

(a) Prove that a pure state $|\psi\rangle_{AB}$ of a two-qubit bipartite system is entangled if and only if its Schmidt number is greater than 1.

(b) Suppose that $|\psi_1\rangle$ and $|\psi_2\rangle$ are two states of a two-qubit bipartite system (with components A and B) having identical Schmidt coefficients. Show that there are unitary transformations U on system A and V on system B such that $|\psi_1\rangle = (U \otimes V) |\psi_2\rangle$.