

# Quantum Cyber Security

## Lecture 2: Quantum Information Basics I

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# Motivation: From Bit-strings to Qubit-strings

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- **Registers** consists of **strings of qubits**

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- **Operations (Gates)** and **Observables** are **linear maps** (matrices): E.g.  $H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{xy} |y\rangle$

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- Measurements are **probabilistic**: The **coefficients determine the probability**. E.g.  $|\psi\rangle = \sum_{x \in \{0,1\}^n} a_x |x\rangle$ , then  $x$  occurs with probability  $|a_x|^2$

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- Multi-qubit operations can generate “**entanglement**”: system behaves “**holistically**” (non-locally – see later)
- **Q**: Why we have speed-up?  
**A**: Like classical probabilistic algorithms BUT with **complex** “probabilities”

A Qubit is a 2-dimensional unit vector

- For formal definitions look at: Math Supplement; Nielsen & Chuang; or first lectures of IQC (<https://opencourse.inf.ed.ac.uk/iqc/course-materials/schedule> or an older version <http://pwallden.gr/courseiqc.asp>)

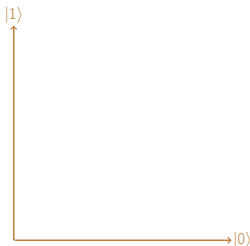


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- We will denote a vector  $\vec{v}$  as  $|v\rangle$

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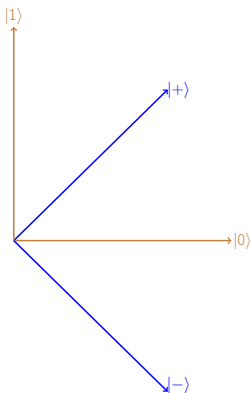
- The unit vectors in the  $x$ -axis as  $|0\rangle$  and in the  $y$ -axis as  $|1\rangle$



# Definitions with Examples

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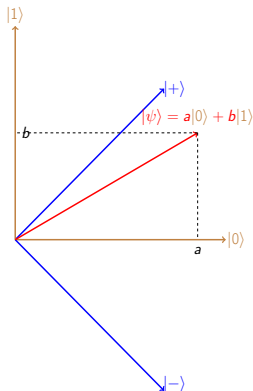
- Another basis (45° rotated) is given by the vectors  $\{|+\rangle, |-\rangle\}$ , where  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ ;  $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$



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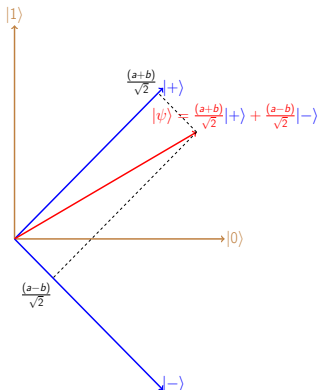
- General Qubit:  $|\psi\rangle = a|0\rangle + b|1\rangle$  where  $||\psi\rangle|^2 = 1 = |a|^2 + |b|^2$  and  $a, b$  are complex numbers in general



# Definitions with Examples

A Qubit is a 2-dimensional unit vector

- Can be expressed in the blue basis:  $|\psi\rangle = \frac{(a+b)}{\sqrt{2}}|+\rangle + \frac{(a-b)}{\sqrt{2}}|-\rangle$



- **Vector** (notation)  $|\psi\rangle$  is called “**ket**”.  
Example:  $|\psi\rangle = a|0\rangle + b|1\rangle$
- **Dual vector** is denoted  $\langle\psi|$  and is called “**bra**”. Coefficients are complex conjugate of the coefficients of the vectors  
Example:  $\langle\psi| = a^* \langle 0| + b^* \langle 1|$
- **Inner product** (c.f. dot-product) is taken between a **vector** and a **dual vector** (c.f. “**bra-ket**”).

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- **Inner product** (c.f. dot-product) is taken between a **vector** and a **dual vector** (c.f. “**bra-ket**”).
- Orthogonal vectors have zero inner product so:  
 $\langle 0|1\rangle = \langle 1|0\rangle = 0$  and  $\langle 0|0\rangle = \langle 1|1\rangle = 1$
- Example:  $\langle\psi_2|\psi_1\rangle = a_2^*a_1 + b_2^*b_1 = \langle\psi_1|\psi_2\rangle^*$   
Let  $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  ;  $|\psi_2\rangle = \frac{1}{2}(i|0\rangle + \sqrt{3}|1\rangle)$   
Check:  $\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1$  and  
 $\langle\psi_2|\psi_1\rangle = \frac{\sqrt{3}-i}{2\sqrt{2}}$  ;  $\langle\psi_1|\psi_2\rangle = \frac{\sqrt{3}+i}{2\sqrt{2}}$

In matrix notation:

**Vectors:**  $|\psi_1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$  and **Dual Vectors:**  $\langle\psi_2| = (a_2^* \quad b_2^*)$



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- Operations (gates) and Observables correspond to **linear maps**

(Complex valued) Matrix with matrix elements  $m_{ij}$

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} = \sum_{i,j \in \{0,1\}} m_{ij} |i\rangle \langle j|$$

- **Outer Product** between a vector and a dual vector (opposite order of inner “**ket-bra**”):

$$|\psi_1\rangle \langle\psi_2| = \begin{pmatrix} a_1 a_2^* & a_1 b_2^* \\ b_1 a_2^* & b_1 b_2^* \end{pmatrix}$$

Example:  $A = \begin{pmatrix} 1 & 1+i \\ 2 & 3+2i \end{pmatrix} = |0\rangle\langle 0| + (1+i)|0\rangle\langle 1| + 2|1\rangle\langle 0| + (3+2i)|1\rangle\langle 1|$

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- **Adjoint** (Hermitian conjugate) of an operator is defined as: **transpose and conjugate element-wise**

Example:  $A^\dagger = \begin{pmatrix} 1 & 2 \\ 1-i & 3-2i \end{pmatrix}$  Note:  $|v\rangle^\dagger = \langle v|$  and

$$(A|v\rangle)^\dagger = \langle v|A^\dagger \text{ and}$$

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Example: The matrix  $A$  above is NOT Hermitian, while the matrix  $B$  is

$$B = \begin{pmatrix} 1 & 2+3i \\ 2-3i & 5 \end{pmatrix} = B^\dagger$$

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These operators, restrict/project a vector to some subspace of the total Hilbert space

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Unitary operators preserve the inner product of vectors  
 $\langle v|w\rangle = \langle v|U^\dagger U|w\rangle$

- **Operations/gates/channels** for (pure) quantum states are unitaries and they map quantum states to quantum states  $U|\psi\rangle = |\phi\rangle$  noting that  $\langle\phi|\phi\rangle = 1 = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle$

**Examples:** Identity **I**; Pauli **X**, **Y** and **Z** gates

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hadamard **H**

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

## Example:

- The quantum NOT-gate is the Pauli  $X$ :

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Acts as the NOT-gate to computational basis vectors:

$$|0\rangle \rightarrow |1\rangle \text{ and } |1\rangle \rightarrow |0\rangle$$

For a general qubit:  $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|1\rangle + \beta|0\rangle$

$$\alpha|0\rangle + \beta|1\rangle \text{ — } \boxed{X} \text{ — } \alpha|1\rangle + \beta|0\rangle$$

- **Measurement** (projective) for pure states
- Computational basis: Given the state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  we measure in the  $\{|0\rangle, |1\rangle\}$  basis
  - With probability  $|\alpha|^2$  we get the outcome **0**; output state is  $|0\rangle$
  - With probability  $|\beta|^2$  we get the outcome **1**; output state is  $|1\rangle$
- General basis: We express the state in that basis and repeat

**Example:** To measure in the  $\{|+\rangle, |-\rangle\}$  basis we re-express  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  in that basis:

$$|\psi\rangle = \frac{(a+b)}{\sqrt{2}}|+\rangle + \frac{(a-b)}{\sqrt{2}}|-\rangle$$

- Outcome **+** with prob  $|\frac{(a+b)}{\sqrt{2}}|^2$  and final state  $|+\rangle$
- Outcome **-** with prob  $|\frac{(a-b)}{\sqrt{2}}|^2$  and final state  $|-\rangle$

# Definitions with Examples

- **Check:** What happens if one measures  $|+\rangle$  in the  $\{|0\rangle, |1\rangle\}$  and in the  $\{|+\rangle, |-\rangle\}$  bases?
- Measurement formally: Given two projection  $P_1, P_2$  where  $P_1 + P_2 = I$ 
  - Outcome cor. to  $P_1$  with probability  $\langle\psi| P_1 |\psi\rangle$  and output state  $(P_1 |\psi\rangle) \frac{1}{\sqrt{\langle\psi| P_1 |\psi\rangle}}$
  - Outcome cor. to  $P_2$  with probability  $\langle\psi| P_2 |\psi\rangle$  and output state  $(P_2 |\psi\rangle) \frac{1}{\sqrt{\langle\psi| P_2 |\psi\rangle}}$
- Note: the sum of probabilities is one:

$$\begin{aligned}\langle\psi| P_1 |\psi\rangle + \langle\psi| P_2 |\psi\rangle &= \langle\psi| (P_1 |\psi\rangle + P_2 |\psi\rangle) = \\ &= \langle\psi| (P_1 + P_2) |\psi\rangle = \langle\psi| I |\psi\rangle = 1\end{aligned}$$

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The trace of an operator is invariant under unitary similarity transformations  $A \rightarrow UAU^\dagger$

$$\text{Tr}(UAU^\dagger) = \text{Tr}(U^\dagger UA) = \text{Tr}(A)$$

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- **Definition**: Assume that the (real) quantum state is one of a number of states  $\{|\psi_i\rangle\}_i$ , each of them occurring with probability  $p_i$ . We call  $\{p_i, |\psi_i\rangle\}$  an **ensemble of states**.

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The state of this system is described by the following density matrix:  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$



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  - 1 **Classical randomness** since we do not know which is the (real) pure quantum state. This randomness is due to the lack of knowledge that we (the observers) have. Is the same with the randomness of classical physics (epistemic).
  - 2 Fundamental **quantum randomness**. This is due to the fact that even if we know the exact pure quantum state (have maximum information about the system), multiple outcomes may occur.

# Example: Classical Vs Quantum Randomness

- Mixed state:

$$\rho_1 = 1/2 |0\rangle \langle 0| + 1/2 |1\rangle \langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

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- Measured in computational basis  $\{|0\rangle, |1\rangle\}$  both give same probabilities (but for  $\rho_1$  is classical randomness while for  $\rho_2$  is quantum randomness).
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- Difference between **maximally mixed** and **equal superposition!**

**Definition:** A **density matrix** is a matrix (or operator)  $\rho$  that:

- 1 is Hermitian  $\rho^\dagger = \rho$
- 2 positive semi-definite (i.e. has non-negative eigenvalues)
- 3 has unit trace  $\text{Tr}(\rho) = 1$



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Exercise: Check that these conditions are satisfied

- 1 for pure density matrices
- 2 for density matrices of the form  $\rho = \sum_i p_i |\psi\rangle \langle \psi|$

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Example:  $\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$

**Ensemble 1:**  $\{p(0) = 3/4, |0\rangle, p(1) = 1/4, |1\rangle\}$

**Ensemble 2:**  $\{p(a) = 1/2, |a\rangle, p(b) = 1/2, |b\rangle\}$  where

$$|a\rangle = \sqrt{\frac{3}{4}} |0\rangle + \sqrt{\frac{1}{4}} |1\rangle$$

$$|b\rangle = \sqrt{\frac{3}{4}} |0\rangle - \sqrt{\frac{1}{4}} |1\rangle$$

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Ensemble 2:  $\{p(a) = 1/2, |a\rangle, p(b) = 1/2, |b\rangle\}$  where

$$|a\rangle = \sqrt{\frac{3}{4}} |0\rangle + \sqrt{\frac{1}{4}} |1\rangle$$

$$|b\rangle = \sqrt{\frac{3}{4}} |0\rangle - \sqrt{\frac{1}{4}} |1\rangle$$

Check that:  $\rho = \frac{1}{2} |a\rangle \langle a| + \frac{1}{2} |b\rangle \langle b| = \frac{3}{4} |0\rangle \langle 0| + \frac{1}{4} |1\rangle \langle 1|$

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- More information will be given in later lectures.
- **Operations:**  $\rho \rightarrow U\rho U^\dagger$ ; norm same  $\text{Tr}(U\rho U^\dagger) = \text{Tr}(\rho) = 1$

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- **Measurements:** Projective measurement  $P_1, P_2$ , at state  $\rho$ .
  - Probability of outcomes  $p_1 = \text{Tr}(P_1\rho)$  ;  $p_2 = \text{Tr}(P_2\rho)$
  - State after measurement

$$\rho_1 = P_1\rho P_1 \frac{1}{\text{Tr}(P_1\rho)} ; \rho_2 = P_2\rho P_2 \frac{1}{\text{Tr}(P_2\rho)}$$

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- Possible values of measuring the observable are the **eigenvalues**
- Probability of each outcome is given by **projecting on the corresponding eigenspace**