Introduction to Quantum Programming and Semantics: Dual objects

Chris Heunen

Spring 2024

Dualizability is a property of an object that means the wire representing it in the graphical calculus can bend. In terms of linear algebra, it is a categorical model for entangled states.

3.5 Compact categories

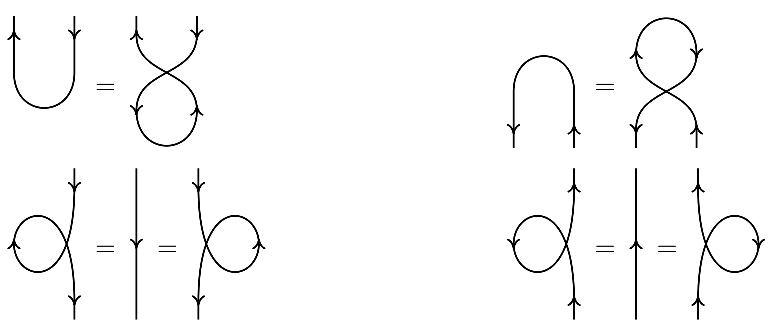
We will be interested in symmetric monoidal categories with duals.

Definition 3.66. A *compact category* is a symmetric monoidal category where every object has dual.

Example 3.67. Since they are symmetric monoidal categories with duals, our main example categories **FHilb**, **FVect**, $\mathbf{Mat}_{\mathbb{C}}$, **Rel** can all be considered compact categories.

When using the graphical calculus, there is now an extra *orientation* on the wires. The following lemma shows that we need not be careful with loops on a single strand.

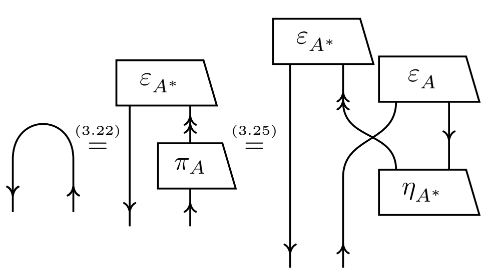
Lemma 3.68. *In a compact category, the following equations hold:*

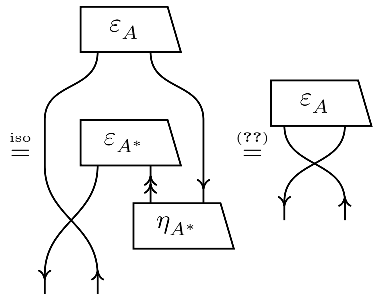


(3.74)

(3.75)

Proof. Let's prove the second equation of (3.74):



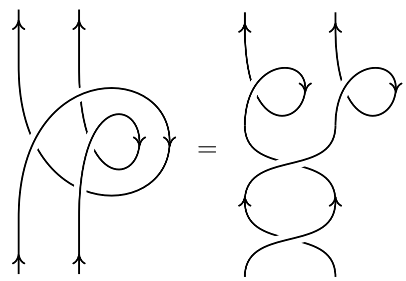


The others are proved in a similar way. □

Here is the correctness theorem making precise that we don't need to be too careful graphically.

Theorem 3.69 (Correctness of the graphical calculus for compact categories)**.** *A well-formed equation between morphisms in a compact category follows from the axioms if and only if it holds in the graphical language up to oriented isotopy in four dimensions.*

We could have got by with a bit weaker structure than symmetric monoidal with chosen duals for this theorem. Framed isotopy is the name for the version of isotopy where the strands are thought of as ribbons, rather than just wires. To get a feeling for framed isotopy, find some ribbons, or make some by cutting long, thin strips from a piece of paper. Use them to verify the following equation:



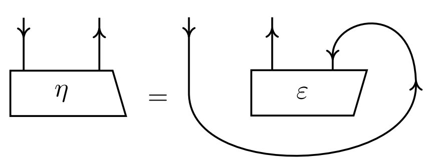
(3.76)

Dagger duality

Lemma 3.70. *In a monoidal dagger category,* $L \dashv R \Leftrightarrow R \dashv L$*.*

Proof. Follows directly from the axiom $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ of a monoidal dagger category. □

Definition 3.71. In a dagger category that is also a compact category, a *dagger dual* is a duality $A \dashv A^*$ witnessed by morphisms $I \xrightarrow{\eta} A^* \otimes A$ and $A \otimes A^* \xrightarrow{\varepsilon} I$, satisfying the following condition:



(3.77)

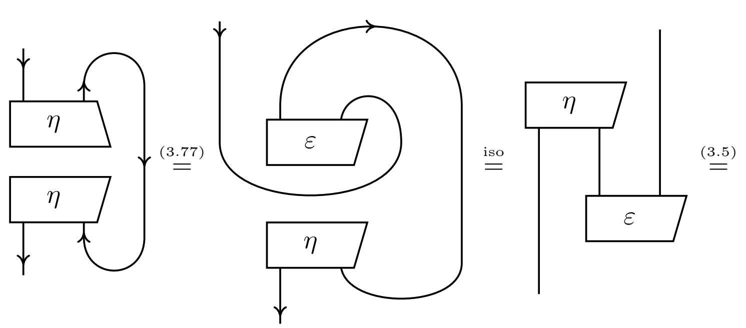
A *compact dagger category* is a symmetric monoidal dagger category whose every object has a dagger dual.

Definition 3.72. In a compact dagger category, a *maximally entangled state* is a bipartite state satisfying the following equations:

(3.78)

Lemma 3.73. *In a compact dagger category, a bipartite state is maximally entangled if and only if it is part of a dagger duality.*

Proof. Use the dagger dual condition (3.77) to verify the first equation of (3.78):

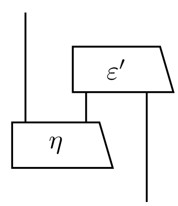


(3.79)

The central isotopy here is a bit hard to see; the box ε makes a full rotation. The other equation, and the reverse implication, can be proved in a similar way. □

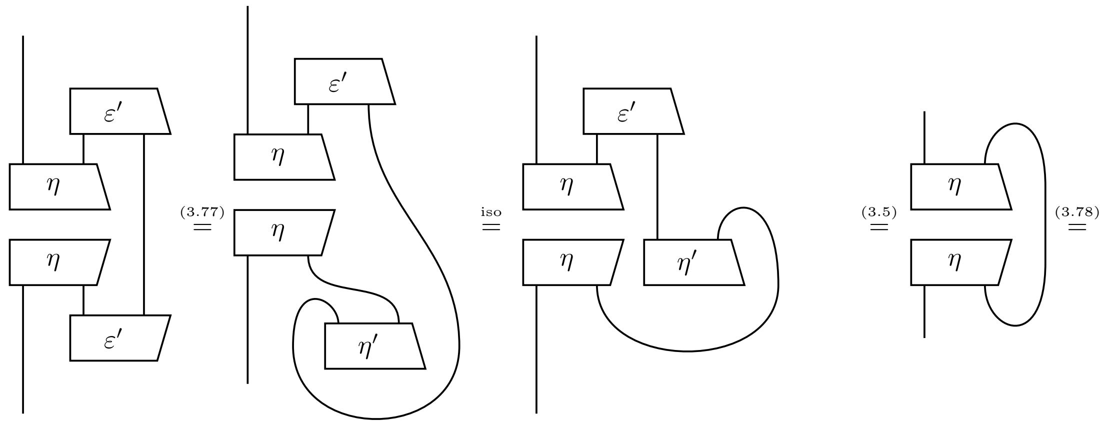
Lemma 3.74. *In a compact dagger category, dagger duals are unique up to unique unitary isomorphism.*

Proof. Given dagger duals $(L \dashv R, \eta, \varepsilon)$ and $(L \dashv R', \eta', \varepsilon')$, we construct an isomorphism $R \simeq R'$ as for Lemma 3.4 as follows:



(3.80)

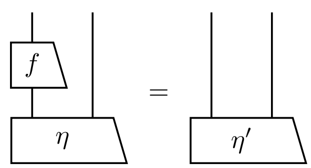
The following calculation establishes that this is a co-isometry:



As with the previous proof, the central isotopy here is a bit tricky to see; the η' morphism performs a full anticlockwise rotation. Similarly, it can be shown that equation (3.80) is also an isometry, and hence unitary. Uniqueness is straightforward. ☐

Putting the previous results together proves the following theorem about maximally-entangled states.

Theorem 3.75. *In a compact dagger category, for any two maximally entangled states* $I \xrightarrow{\eta,\eta'} A \otimes B$ *there is a unique unitary* $A \xrightarrow{f} A$ *satisfying the following equation:*



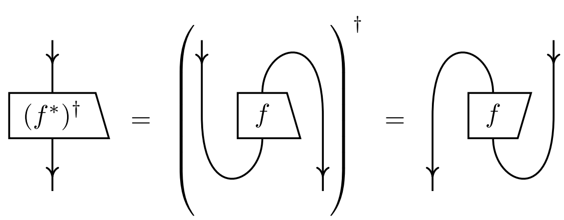
(3.81)

Proof. This follows from Lemmas 3.73 and 3.74. ☐

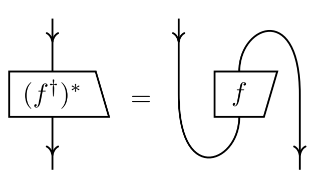
Lemma 3.76. *In a compact dagger category, every morphism* f *satisfies the following equation:*

$$(f^*)^\dagger = (f^\dagger)^* \tag{3.82}$$

Proof. Compute both sides:



(3.83)



(3.84)

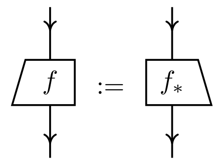
These are isotopic, and hence equal by the correctness theorem. ☐

Definition 3.77. On a compact dagger category, *conjugation* $(-)_*$ is defined as the composite of the dagger functor and the right-duals functor:

$$(-)_* := (-)^{*\dagger} = (-)^{\dagger *} \tag{3.85}$$

Since taking daggers is the identity on objects we have $A_* := A^*$. Also, since $(-)^*$ and taking daggers are both contravariant, the conjugation functor is covariant.

We denote conjugation graphically by flipping the morphism box about a vertical axis:



(3.86)

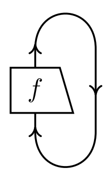
Example 3.78. Our examples **FHilb**, $\mathbf{Mat}_{\mathbb{C}}$ and **Rel** are all compact dagger categories.

- On **FHilb**, the conjugation functor gives the conjugate of a linear map.
- On $\mathbf{Mat}_{\mathbb{C}}$, the conjugation functor gives the conjugate of a matrix, with each matrix entry replaced by its conjugate as a complex number.
- On **Rel**, the conjugation functor is the identity.

3.6 Traces and dimensions

Square matrices have an important construction, the trace, which plays a fundamental role in linear algebra. In this section we see how traces arise categorically in compact categories.

Definition 3.79 (Trace)**.** In a compact dagger category, the *trace* of a morphism $A \xrightarrow{f} A$ is the following scalar:



(3.87)

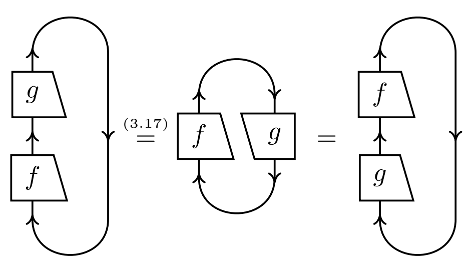
It is denoted by $\mathrm{Tr}(f)$, or sometimes $\mathrm{Tr}_A(f)$ to emphasize A. (Don't confuse it with the partial trace of quantum theory.)

Definition 3.80. In a compact dagger category, the *dimension* of an object A is the scalar $\dim(A) := \mathrm{Tr}(\mathrm{id}_A)$.

This abstract trace operation, like its concrete cousin from linear algebra, enjoys the familiar cyclic property.

Lemma 3.81. *In a compact dagger category, morphisms* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} A$ *satisfy* $\mathrm{Tr}_A(g \circ f) = \mathrm{Tr}_B(f \circ g)$.

Proof. We can show this graphically in the following way:



(3.88)

The morphism g slides around the circle, and ends up underneath the morphism f. □

Example 3.82. To determine $\mathrm{Tr}(f)$ for a morphism $H \xrightarrow{f} H$ in **FHilb**, substitute equations (3.7) and (3.6) into the definition of the abstract trace (3.87). Then $\mathrm{Tr}(f) = \sum_i \langle i|f|i\rangle$, so the abstract trace of f is in fact the usual trace of f from linear algebra. Therefore, for an object H of **FHilb**, also $\dim(H) = \mathrm{Tr}(\mathrm{id}_H)$ equals the usual dimension of H.

Abstract traces satisfy many properties familiar from linear algebra.

Lemma 3.83. *In a compact dagger category, the trace has the following properties:*

(a) $\mathrm{Tr}_I(s) = s$*;*

(b) $\mathrm{Tr}_{A\otimes B}(f \otimes g) = \mathrm{Tr}_A(f) \circ \mathrm{Tr}_B(g)$ *in a compact category;*

(c) $\big(\mathrm{Tr}_A(f)\big)^\dagger = \mathrm{Tr}_A(f^\dagger)$ *in a compact dagger category.*

Proof. Property (a) follows from $\mathrm{Tr}_I(s) = s \bullet \mathrm{id}_I = s$, which trivializes graphically. Property (b) follows because the traces over A and B can separate in a braided monoidal category; the inner one is not trapped by the outer one. Finally, property (c) follows from correctness of the graphical language for compact dagger categories. □

This immediately yields some properties of dimensions of objects.

Lemma 3.84. *In a compact dagger category, the following properties hold:*

(a) $\dim(I) = \mathrm{id}_I$*;*

(b) $\dim(A \otimes B) = \dim(A) \circ \dim(B)$*.*

(c) $A \simeq B \Rightarrow \dim(A) = \dim(B)$*;*

Proof. Properties (a) and (b) are straightforward consequences of Lemma 3.83. Property (c) follows from the cyclic property of the trace demonstrated in Lemma 3.81: if $A \xrightarrow{k} B$ is an isomorphism, then $\dim(A) = \mathrm{Tr}_A(k^{-1} \circ k) = \mathrm{Tr}_B(k \circ k^{-1}) = \dim(B)$. □

In a similar way, we can prove that if a category had coinciding products and coproducts (like the direct sum of Hilbert spaces), then $\mathrm{Tr}_{A\oplus B}\begin{pmatrix} f & g \\ h & j \end{pmatrix} = \mathrm{Tr}_A(f) + \mathrm{Tr}_B(j)$. This gives a simple argument that infinite-dimensional Hilbert spaces cannot have duals.

Corollary 3.85. *Infinite-dimensional Hilbert spaces do not have duals.*

Proof. Suppose H is an infinite-dimensional Hilbert space. Then there is an isomorphism $H \oplus \mathbb{C} \simeq H$. If H had a dual, then this would imply $\dim(H) + 1 = \dim(H)$, which has no solutions for $\dim(H) \in \mathbb{C}$. □

This argument would not apply in **Rel**, since we have $\mathrm{id}_1 + \mathrm{id}_1 = \mathrm{id}_1$ in that category. And indeed, as we have seen at the beginning of this chapter, both finite and infinite sets are self-dual in this category, despite the fact that sets S of infinite cardinality can be equipped with isomorphisms $S \simeq S \cup 1$.