Introduction to Quantum Programming and Semantics

Week 9: Complementarity

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informatics

Overview

- Incompatible Frobenius structures: mutually unbiased bases
- Deutsch–Jozsa algorithm: prototypical use of complementarity
- Quantum groups: strong complementarity
- Qubit gates: quantum circuits

Idea

- Measure qubit in basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$, then in $\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\}$: probability of either outcome $1/2$.

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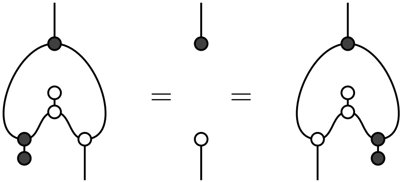
- First measurement provides no information about second: Heisenberg's *uncertainty principle*.
- Orthogonal bases $\{a_i\}$ and $\{b_j\}$ are complementary/unbiased if

$$\langle a_i | b_j \rangle \langle b_j | a_i \rangle = c$$

for some $c \in \mathbb{C}$.

Complementarity

In braided monoidal dagger category, symmetric dagger Frobenius structures  and  on the same object are complementary if:



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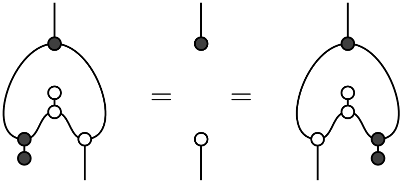
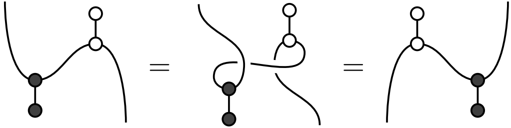


Black and white not obviously interchangeable. But by symmetry:



So could have added two more equalities.

Complementarity in **FHilb**

Commutative dagger Frobenius structures in **FHilb** complementary if and only if they copy complementary bases (with $c = 1$).

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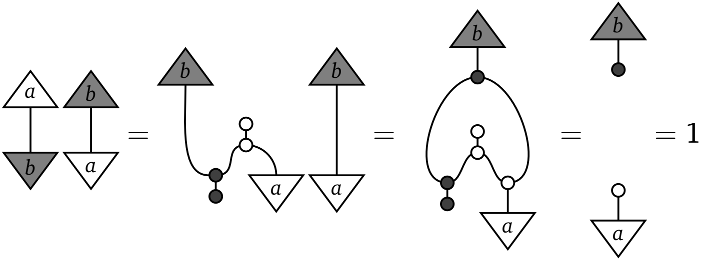
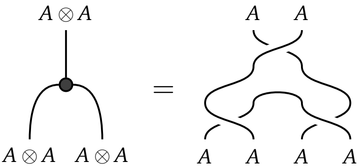
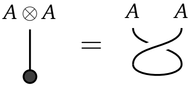
Proof. For all a in white basis, and b in black basis:



Twisted knickers

In compact dagger category, if A is self-dual, the following Frobenius structure on $A \otimes A$ is complementary to pair of pants:





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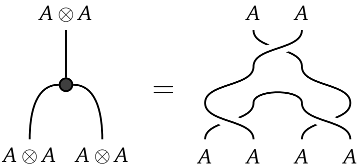

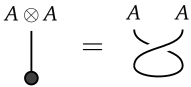

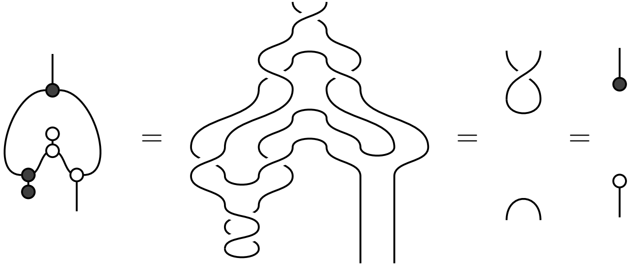

So Frobenius structure on A gives complementary pair on $A \otimes A$.

Pauli basis

Three mutually complementary bases of $\mathbb{C}^2$:

$$
\begin{aligned}
&X \text{ basis} && \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \\
&Y \text{ basis} && \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \\
&Z \text{ basis} && \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}
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- Largest family of complementary bases for $\mathbb{C}^2$: no four bases all mutually unbiased.

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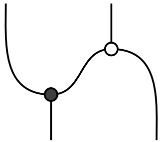
\end{array}
$$

- Largest family of complementary bases for $\mathbb{C}^2$: no four bases all mutually unbiased.
- What is the maximum number of mutually complementary bases in a given dimension? Only known for prime power dimensions p^n.

Characterisation

Symmetric dagger Frobenius structures in braided monoidal dagger category are complementary if and only if the following is unitary:



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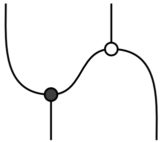
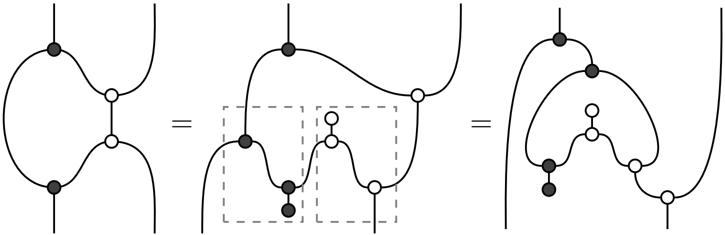


Proof. Compose with adjoint:



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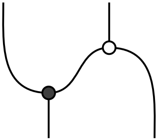

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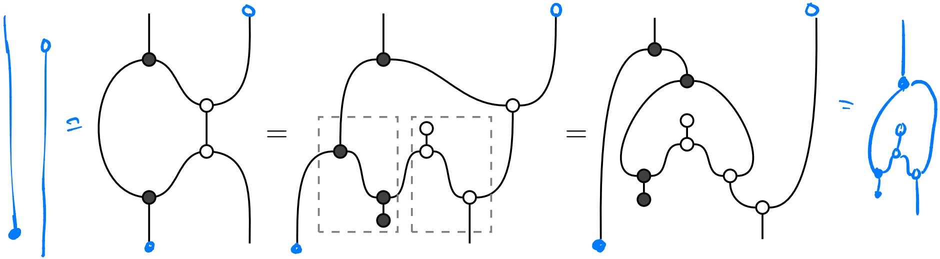

Conversely, if is identity, compose with white counit on top right, black unit on bottom left, to get complementarity.

Complementarity in **Rel**

If G, H are nontrivial groups, these are complementary groupoids:

- objects $g \in G$, morphisms $g \xrightarrow{(g,h)} g$, with $(g, h') \bullet (g, h) = (g, hh')$
- objects $h \in H$, morphisms $h \xrightarrow{(g,h)} h$, with $(g', h) \circ (g, h) = (gg', h)$

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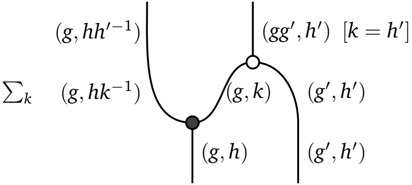

Every input related to unique output, so unitary.

Groupoid allows complementary one just when every object has number of outgoing morphisms.

The Deutsch-Jozsa algorithm

Solves certain problem faster than possible classically

- Typical exact quantum decision algorithm (no approximation)
- Problem artificial, but other important algorithms very similar:
  - Shor's factoring algorithm
  - Grover's search algorithm
  - the hidden subgroup problem
- 'All or nothing' nature makes it categorical

The Deutsch-Jozsa algorithm

Problem:

- Given 2-valued function $A \xrightarrow{f} \{0, 1\}$ on a finite set A.
- Constant if takes just a single value on every element of A.
- Balanced if takes value 0 on exactly half the elements of A.
- You are promised that f is either constant or balanced.
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Best classical strategy:

- Sample f on $\frac{1}{2}|A| + 1$ elements of A.
  If different values then balanced, otherwise constant.

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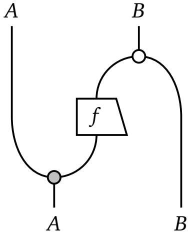
How to access f? Can only apply unitary operators...
Must embed $A \xrightarrow{f} \{0, 1\}$ into an *oracle*.

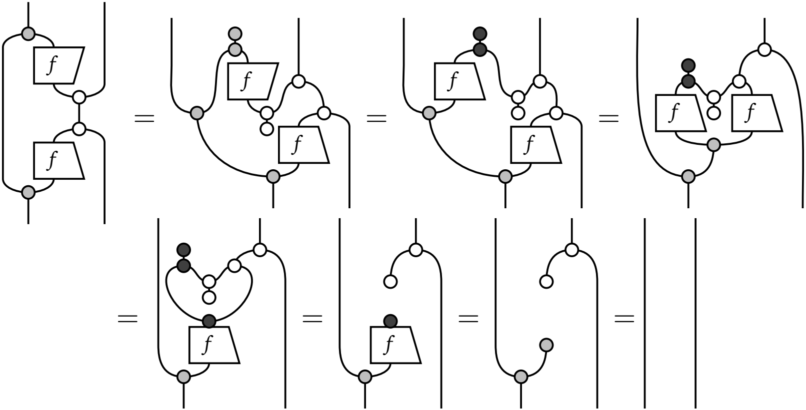
Given Frobenius structures (A, ⅄, ⫯) and (B, ⅄, ⫯) in monoidal dagger category, oracle is morphism $A \xrightarrow{f} B$ making the following unitary:



Where to find oracles

Let (A, ◍), (B, ●) and (B, ○) be symmetric dagger Frobenius.
If ●, ○ complementary, self-conjugate comonoid homomorphism
(A, ◍) $\xrightarrow{f}$ (B, ●) is oracle.

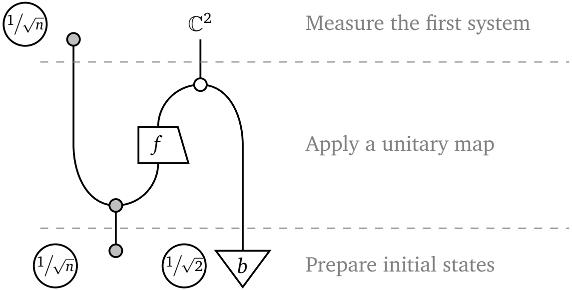
Proof.



The Deutsch-Jozsa algorithm

Let $A \xrightarrow{f} \{0,1\}$ be given function, and $|A| = n$.
Choose complementary bases ● $= \mathbb{C}^2$, ○ $= \mathbb{C}[\mathbb{Z}_2]$.
Let $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, a copyable state of ○.

The Deutsch–Jozsa algorithm is this morphism:



Deutsch-Jozsa simplifies

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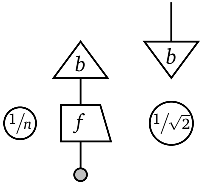

Proof. Duplicate copyable state *b* through white dot, and apply noncommutative spider theorem to cluster of gray dots.

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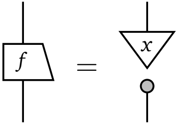

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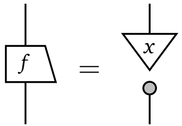
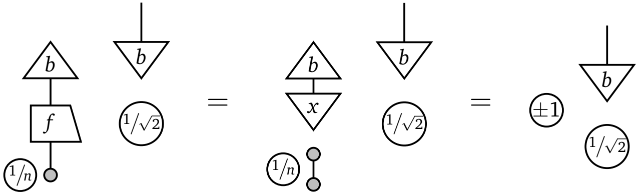


So history is:



This has norm 1, so the history is certain.

Deutsch-Jozsa correctness:balanced

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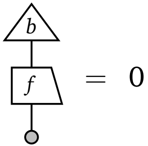
Proof. The function f is balanced just when the following holds:



Recall $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

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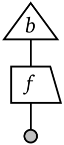

$= \ 0$

Recall $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence the final history equals 0.

Bialgebras

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One standard way: let G be finite group, and consider Hilbert space with basis $\{g \in G\}$, with

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 $: g \mapsto 1$

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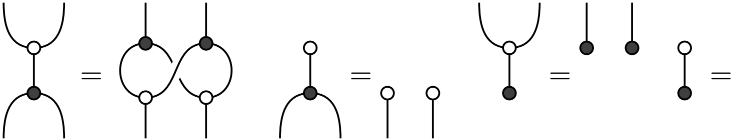
Some nice relationships emerge between  and .

Bialgebras

In a braided monoidal category, a bialgebra consists of a monoid
(A,, ) and a comonoid (A,, ) satisfying:



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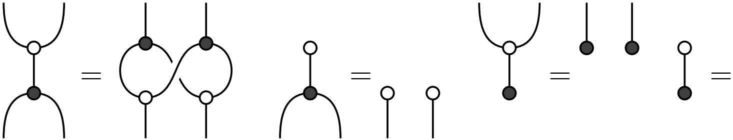

Example: monoid M is a bialgebra in **Set** and hence in **Rel** and **FHilb**

: $m \mapsto (m, m)$ : $m \mapsto \bullet$ : $(m, n) \mapsto mn$ : $\bullet \mapsto 1_M$.

Frobenius hates bialgebras

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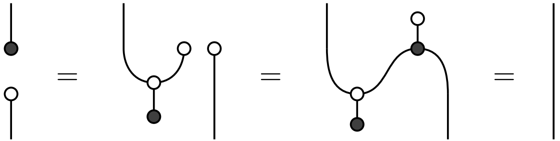
In a braided monoidal category, if a monoid (A,,) and comonoid (A,,) form a Frobenius structure and a bialgebra, then $A \simeq I$.

Proof. Will show  and  are inverses. The bialgebra laws already require  $\circ$  $= \mathrm{id}_I$. For the other composite:



Copyable states

In a braided monoidal category if  and  form a bialgebra, then copyable states for  are a monoid under .

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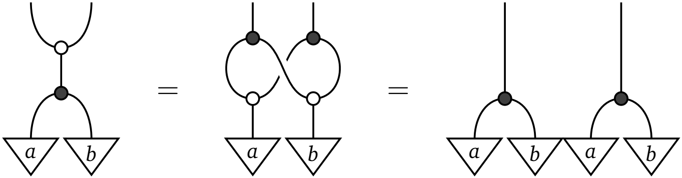
Proof. Associativity is immediate. Unitality comes down to third bialgebra law: is copyable for . Have to prove well-definedness. Let a and b be copyable states for .



Hence -copyable states are indeed closed under .

Strong complementarity

- Consider $\mathbb{C}^2$ in **FHilb**. Computational basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ gives dagger Frobenius structure . Orthogonal basis $\{\begin{pmatrix}e^{i\varphi}\\e^{i\theta}\end{pmatrix}, \begin{pmatrix}e^{i\varphi}\\-e^{i\theta}\end{pmatrix}\}$ gives dagger Frobenius structure . Complementary, but only a bialgebra if $\varphi = \theta = 0$.

Strong complementarity

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- In a braided monoidal dagger category, two dagger symmetric Frobenius structures are strongly complementary when they are complementary, and also form a bialgebra.

Strong complementarity in **FHilb**

In **FHilb**, strongly complementary symmetric dagger Frobenius structures, one of which is commutative, correspond to finite groups.

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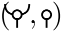
Proof.

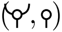
- Given strongly complementary symmetric dagger Frobenius structures, the states that are self-conjugate, copyable and deletable for  form a group under .
- By the classification theorem for commutative dagger Frobenius structures, there is an entire basis of such states for .

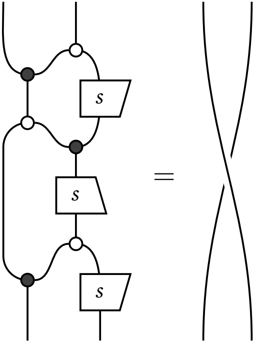
```
f(a,b) {
    bit c := a;
    a := b;
    b := c;

    return (.a, .b);
}
```

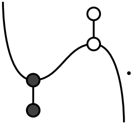
```
                          a = 0    b = 0
a := a XOR b;               0        0
b := b XOR a;               0        0
a := a XOR b;               0        0
```

CNOT(a,b) = (a, a XOR b)

Qubit gates

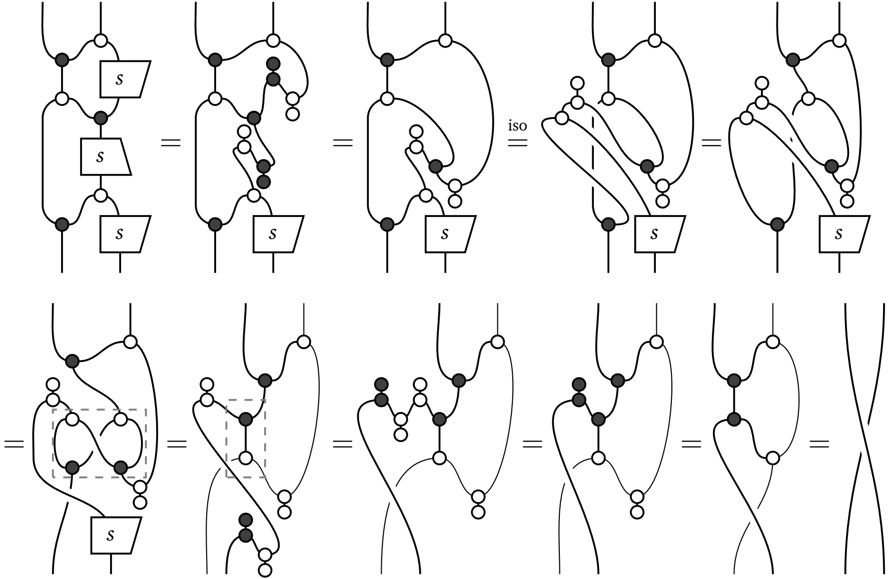
In a braided monoidal dagger category, let  and  be complementary classical structures with antipode s. Then the first bialgebra law holds if and only if:



where $s =$  .

Qubit gates

Proof.



Qubit gates in **FHilb**

Fix A to be qubit $\mathbb{C}^2$; let (,) copy computational basis $\{|0\rangle, |1\rangle\}$, and (,) copy the X basis. The three antipodes s become identities.

The three unitaries reduce to three CNOT gates:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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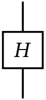
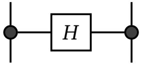
These two classical structures are transported into each other by Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$



Controlled Z

The CZ gate in **FHilb** can be defined as follows.

CZ = 

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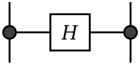
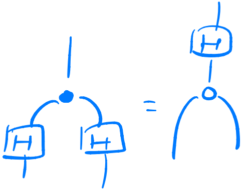
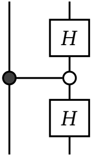
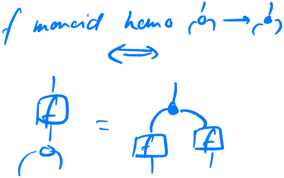




Proof. Rewrite as:

CZ =





Hence

$$\mathrm{CZ} = (\mathrm{id} \otimes H) \circ \mathrm{CNOT} \circ (\mathrm{id} \otimes H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Controlled Z

If $(A,$ $)$ and $(A,$ $)$ complementary classical structures in braided monoidal dagger category, and $A \xrightarrow{H} A$ satisfies $H \circ H = \mathrm{id}_A$, then CZ makes sense and satisfies $\mathrm{CZ} \circ \mathrm{CZ} = \mathrm{id}$.

Controlled Z

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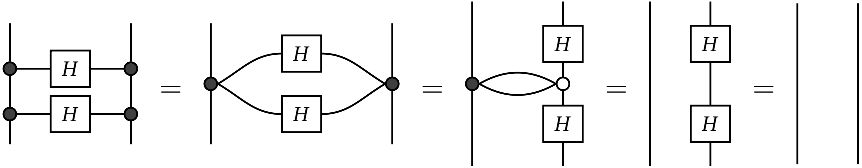
Proof.



Measurement-based computing

Single-qubit unitaries can be implemented via Euler angles: unitary $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$ allows phases φ, ψ, θ with $u = Z_\varphi \circ X_\psi \circ Z_\theta$, where Z_θ is rotation in Z basis over angle θ, and X_φ in X basis over angle φ.

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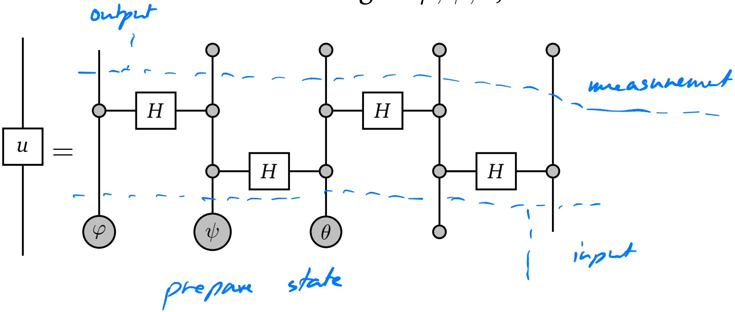
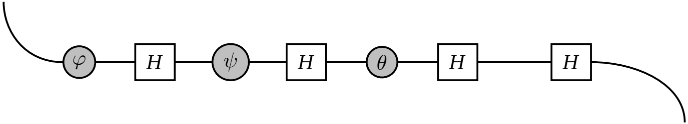
If unitary $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$ in **FHilb** has Euler angles φ, ψ, θ, then:

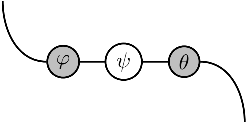


Measurement-based computing

Proof. Use phased spider theorem to reduce to:



But by transport lemma, this is just:



which equals u, by definition of the Euler angles.

Summary

- Incompatible Frobenius structures: mutually unbiased bases
- Deutsch-Jozsa algorithm: prototypical use of complementarity
- Quantum groups: strong complementarity
- Qubit gates: use in quantum circuits