Introduction to Quantum Programming and Semantics

Week 4: Dual objects

Chris Heunen

THE UNIVERSITY of EDINBURGH
informatics

Dual objects

Idea:

- Quantum mechanically: maximally entangled states
- Graphically: bending wires

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An object L is left-dual to an object R, and R is right-dual to L, written $L \dashv R$, when there is a unit morphism $I \xrightarrow{\eta} R \otimes L$ and a counit morphism $L \otimes R \xrightarrow{\varepsilon} I$ such that:

$$
\begin{array}{ccccc}
L & \xrightarrow{\rho_L^{-1}} & L \otimes I & \xrightarrow{\mathrm{id}_L \otimes \eta} & L \otimes (R \otimes L) \\
\Big\downarrow{\scriptstyle \mathrm{id}_L} & & & & \Big\downarrow{\scriptstyle \alpha^{-1}_{L,R,L}} \\
L & \xleftarrow[\lambda_L]{} & I \otimes L & \xleftarrow[\varepsilon \otimes \mathrm{id}_L]{} & (L \otimes R) \otimes L
\end{array}
$$

$$
\begin{array}{ccccc}
R & \xrightarrow{\lambda_R^{-1}} & I \otimes R & \xrightarrow{\eta \otimes \mathrm{id}_R} & (R \otimes L) \otimes R \\
\Big\downarrow{\scriptstyle \mathrm{id}_R} & & & & \Big\downarrow{\scriptstyle \alpha_{R,L,R}} \\
R & \xleftarrow[\rho_R]{} & R \otimes I & \xleftarrow[\mathrm{id}_R \otimes \varepsilon]{} & R \otimes (L \otimes R)
\end{array}
$$

Snake equations

Draw an object L as a wire with an upward-pointing arrow,
and a right dual R as a wire with a downward-pointing arrow.

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Draw an object L as a wire with an upward-pointing arrow, and a right dual R as a wire with a downward-pointing arrow.

The unit $I \xrightarrow{\eta} R \otimes L$ and counit $L \otimes R \xrightarrow{\varepsilon} I$ are drawn as bent wires:

Duality equations become:

Also called the snake equations.

Dual Hilbert spaces

FHilb has all duals: any finite-dimensional Hilbert space H is both right and left dual to its dual Hilbert space H^*, in a canonical way.

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The counit $H \otimes H^* \xrightarrow{\varepsilon} \mathbb{C}$ is:

$$|\phi\rangle \otimes \langle\psi| \mapsto \langle\psi|\phi\rangle$$

The unit $\mathbb{C} \xrightarrow{\eta} H^* \otimes H$ is defined like so, for any orthonormal basis $|i\rangle$:

$$1 \mapsto \sum_i \langle i| \otimes |i\rangle$$

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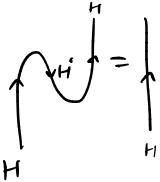
Is η basis-dependent, but ε not? No. (Will prove shortly.)
Infinite-dimensional spaces do not have duals. (Will prove later.)



$$\lambda \circ (\varepsilon \otimes id_H) \circ \alpha \circ (id_H \otimes \eta) \circ \rho^{-1} \; |j\rangle$$

$$= \lambda \circ (\varepsilon \otimes id_H) \circ \alpha \circ (id_H \otimes \eta) \; (|j\rangle \otimes 1)$$

$$= \lambda \circ (\varepsilon \otimes id_H) \circ \alpha \circ \left(|j\rangle \otimes \sum_i \langle i| \otimes |i\rangle\right)$$

$$= \sum_i \lambda \circ (\varepsilon \otimes id_H) \; (|j\rangle \otimes \langle i|) \otimes |i\rangle$$

$$= \sum_i \lambda \left(\langle i|j\rangle \otimes |i\rangle \right)$$

$$= \lambda \left(1 \otimes |j\rangle \right)$$

$$= |j\rangle$$

Dual matrices

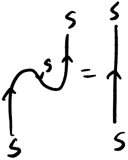
In $\mathbf{Mat}_{\mathbb{C}}$, every object n is its own dual, with a canonical choice of η and ε given as follows:

$$\eta : 1 \mapsto \sum_i |i\rangle \otimes |i\rangle \qquad\qquad \varepsilon : |i\rangle \otimes |j\rangle \mapsto \delta_{ij} 1$$

Dual relations

In **Rel**, every object is its own dual, even infinite sets.
Unit $1 \xrightarrow{\eta} S \times S$ and counit $S \times S \xrightarrow{\varepsilon} 1$ are:

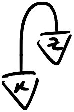
$$\bullet \sim_\eta (s,s) \text{ for all } s \in S$$
$$(s,s) \sim_\varepsilon \bullet \text{ for all } s \in S$$



iff

$\iff x = y$ ✓

$\iff \exists z \in S:$



and

$\iff \exists z \in S: \; x = z$ and $z = y$

$\iff x = y$

$R =$

$: S \to 1$

$T =$

$: 1 \to S$

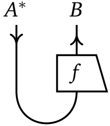
$R \circ T : 1 \to 1$

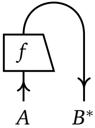
$= \{ (a,c) \mid \exists b : (a,b) \in T, \; (b,c) \in R \}$

Names and conames

Set only has duals for singleton sets.

Let $A \xrightarrow{f} B$ be a morphism in a monoidal category with dualities $A \dashv A^*$ and $B \dashv B^*$. Its name $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes B$ and *coname* $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} I$ are:





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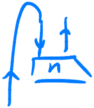
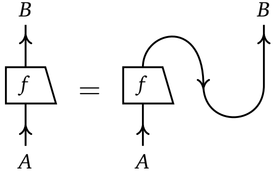
Morphisms can be recovered from their names or conames:

take

$f =$



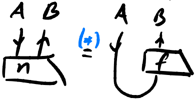


Given $I \xrightarrow{n} A^* \otimes B$

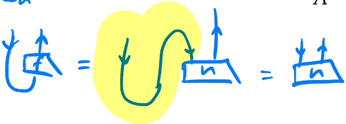
Find $A \xrightarrow{f} B$

such that $n = \ulcorner f \urcorner$.

i.e.



then



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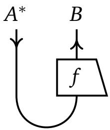
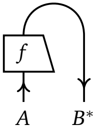
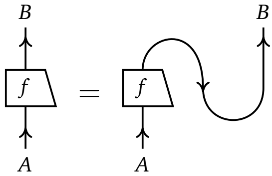




Morphisms can be recovered from their names or conames:



In **Set** I is terminal, and so all conames $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} I$ must be equal. If **Set** had duals this would imply all functions $A \to B$ were equal.

Duals are unique up to iso

In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$.
Similarly, if $L \dashv R$, then $L' \dashv R$ if and only if $L \simeq L'$.

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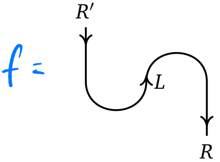
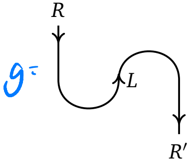
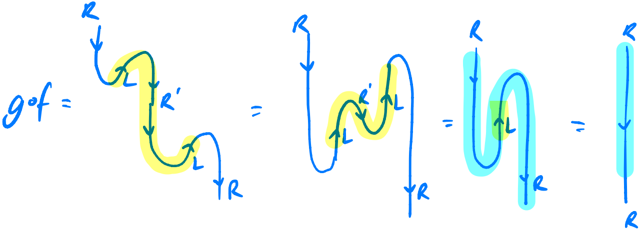
Proof: If $L \dashv R$ and $L \dashv R'$, define maps $R \to R'$ and $R' \to R$ by:





The snake equations imply that these are inverse.



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In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$.
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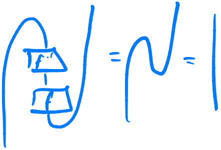
Proof: If $L \dashv R$ and $L \dashv R'$, define maps $R \to R'$ and $R' \to R$ by:

The snake equations imply that these are inverse. Conversely, if $L \dashv R$ and $R \xrightarrow{f} R'$ is invertible, we can construct a duality $L \dashv R'$:



An iso $L \simeq L'$ lets us produce duality $L' \dashv R$ in a similar way.

Unit determines counit

If $(L, R, \eta, \varepsilon)$ and $(L, R, \eta, \varepsilon')$ both exhibit duality, then $\varepsilon = \varepsilon'$.
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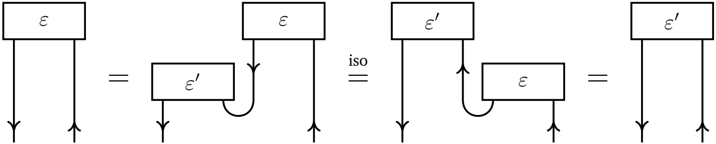

Duals respect tensors

In a monoidal category, $I \dashv I$, and $L \otimes L' \dashv R \otimes R'$ if $L \dashv R$ and $L' \dashv R'$.

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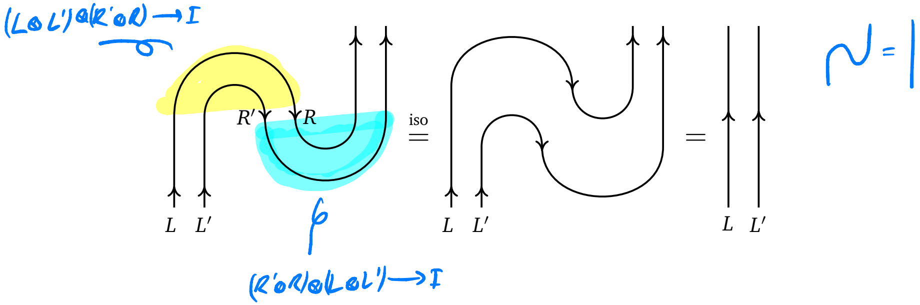
Now suppose $L \dashv R$ and $L' \dashv R'$. We make the new unit and counit maps from the old ones, and compute as follows:



Duals respect braiding

In a braided monoidal category, $L \dashv R \Rightarrow R \dashv L$.

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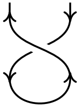
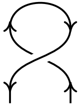
Construct a new duality as follows:



$$I \xrightarrow{\eta'} L \otimes R$$



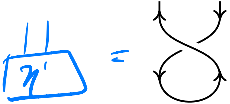
$$R \otimes L \xrightarrow{\varepsilon'} I$$

Duals respect braiding

$L \dashv L^*$

In a braided monoidal category, $L \dashv R \;\Rightarrow\; R \dashv L$.

Construct a new duality as follows:



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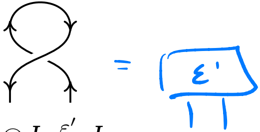
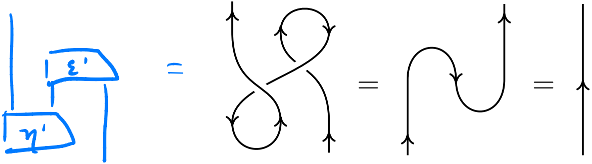


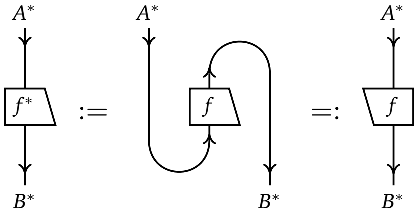
$R \otimes L \xrightarrow{\varepsilon'} I$

Test the snake equations:



Duals for morphisms

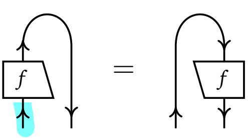
For a morphism $A \xrightarrow{f} B$ and chosen dualities $A \dashv A^*$, $B \dashv B^*$, the right dual $B^* \xrightarrow{f^*} A^*$ is defined in the following way:

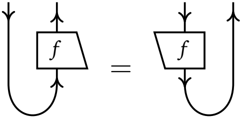


Represent this graphically by rotating the box for f.

Sliding

For all morphisms $A \xrightarrow{f} B$ in a monoidal category with chosen duals $A \dashv A^*$ and $B \dashv B^*$:

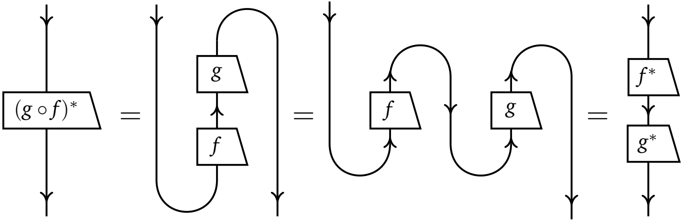




Duals are functorial

If a monoidal category has chosen right duals, $(-)^*$ is a functor.

Proof: Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$.



Similarly, $(\mathrm{id}_A)^* = \mathrm{id}_{A^*}$ follows from the snake equations.

Examples

- In **FVect** and **FHilb**, right dual of $V \xrightarrow{f} W$ is $W^* \xrightarrow{f^*} V^*$, acting as $f^*(e) := e \circ f$, where $W \xrightarrow{e} \mathbb{C}$ is an arbitrary element of W^*.
- In $\mathbf{Mat}_{\mathbb{C}}$, the dual of a matrix is its transpose.
- In **Rel**, the dual of a relation is its converse. So the right duals functor and the dagger functor have the same action: $R^* = R^\dagger$ for all relations R.

Double duals

In monoidal category with chosen right duals, $A^{**} \otimes B^{**} \simeq (A \otimes B)^{**}$.

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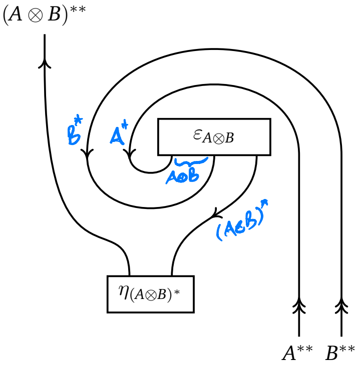
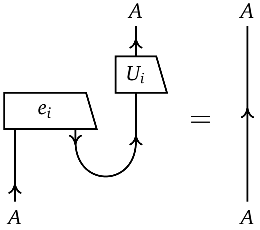
Proof:



Teleportation

In a monoidal category with right duals, a teleportation procedure isa finite family of effects $e_i : A \otimes A^* \to I$ and unitaries $U_i : A \to A$ with:



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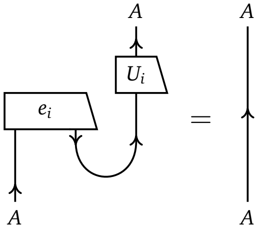
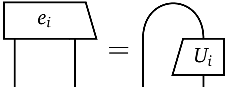


This can be solved to give 

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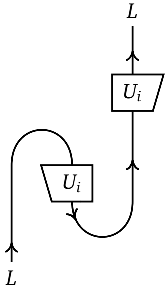
Teleportation

Simplify the history:



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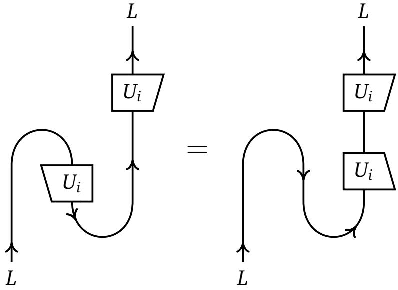

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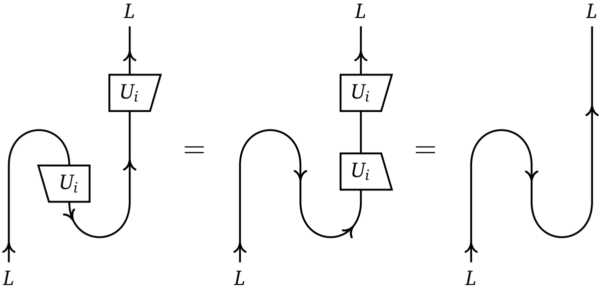

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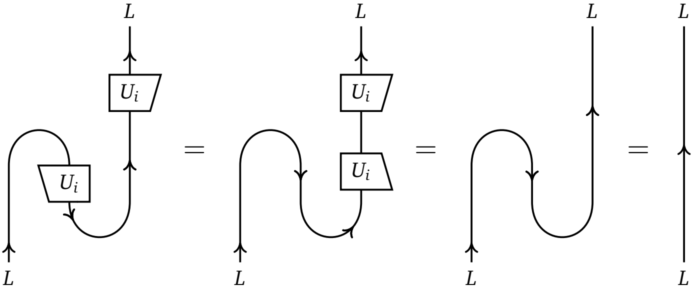

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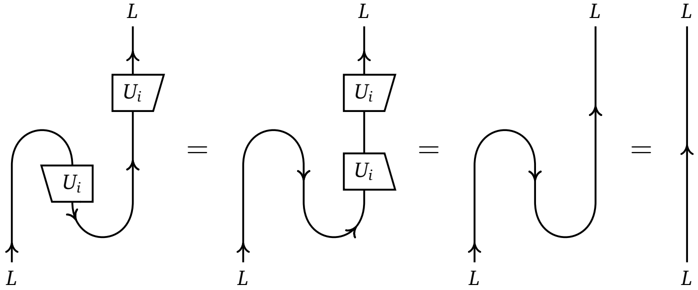

So if the original history occurs, the result is for the state of the original system to be transmitted faithfully.

If $\{e_i\}$ is a *complete* set of effects, this will always succeed.

Teleportation in **Hilb**

Choose $L = R = \mathbb{C}^2$ and $\eta^\dagger = \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$, and unitaries U_i:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This gives rise to the following family of effects:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}$$

This is a complete set of effects, since it forms a basis for the vector space $\mathbf{Hilb}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C})$. So it is guaranteed to be successful.

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This is traditional qubit teleportation.

Teleportation in **Rel**

Choose $L = R = \{0, 1\}$ and $\eta^\dagger = \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$, and unitaries:

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This gives rise to the following family of effects:

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These form a complete set of effects.

Teleportation in **Rel**

Choose $L = R = \{0, 1\}$ and $\eta^\dagger = \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$, and unitaries:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This gives rise to the following family of effects:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$$

These form a complete set of effects.

This is classical encrypted communication with a one-time pad.

Graphical calculus for compact categories

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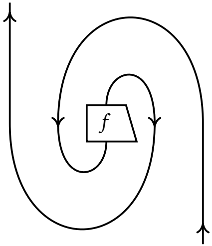

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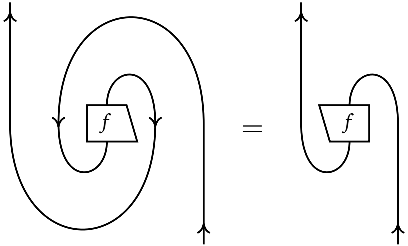

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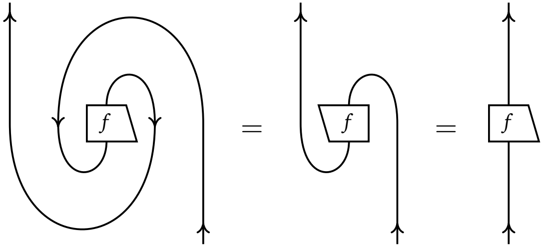

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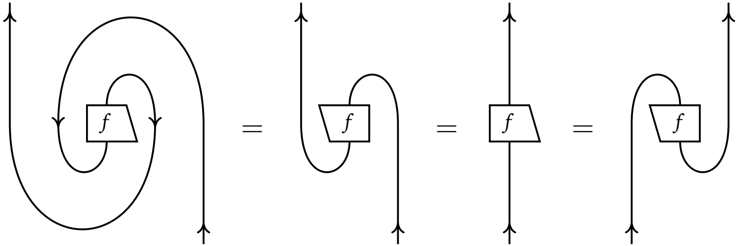
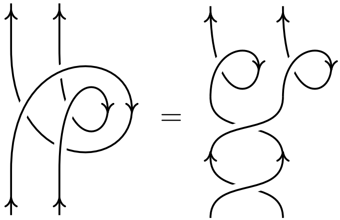
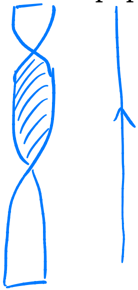


Intermezzo: ribbon categories

Could have got by with less than symmetric monoidal with duals.
Useful in topological quantum computation.

Make some ribbons by cutting long, thin strips from piece of paper.
Verify:





Compact dagger categories

In a monoidal dagger category, $L \dashv R \Leftrightarrow R \dashv L$.
Proof: follows directly from axiom $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$.

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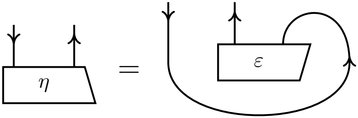
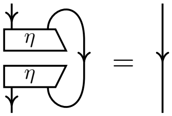
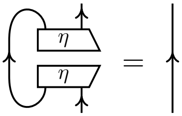
In a monoidal dagger category, a dagger dual is a duality $A \dashv A^*$ witnessed by morphisms $I \xrightarrow{\eta} A^* \otimes A$ and $A \otimes A^* \xrightarrow{\varepsilon} I$ satisfying:



Maximally entangled states

In a compact dagger category, a maximally entangled state is a bipartite state with:





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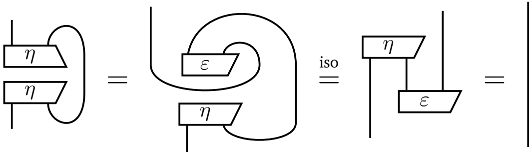
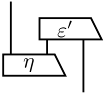
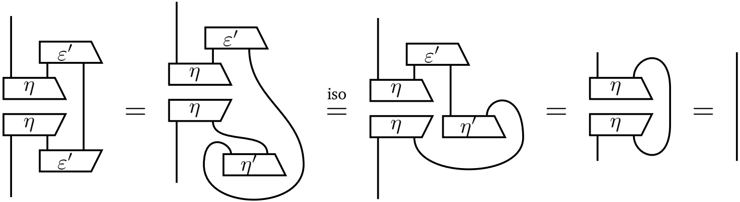


Dagger duals unique up to unitary

Given dagger duals $(L \vdash R, \eta, \varepsilon)$ and $(L \vdash R', \eta', \varepsilon')$, construct an isomorphism $R \simeq R'$ as before:

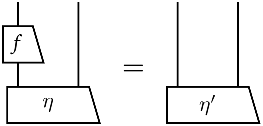


Then:



Maximally entangled states unique up to unitary

In a compact dagger category, any two maximally entangled states $I \xrightarrow{\eta,\eta'} A \otimes B$ have a unique unitary $A \xrightarrow{f} A$ with:

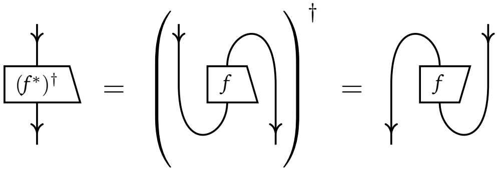


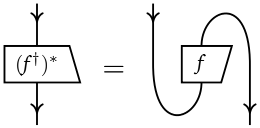
So maximally entangled states are unique up to a unique unitary.

Conjugation

In a compact dagger category, every morphism satisfies $(f^*)^\dagger = (f^\dagger)^*$.

Proof:





Conjugation

On a compact dagger category, conjugation $(-)_*$ is defined as the composite of the dagger and the right-duals functor:

$$(-)_* := (-)^{*\dagger} = (-)^{\dagger *}$$

Since taking daggers is the identity on objects we have $A_* := A^*$.

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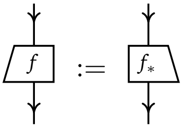
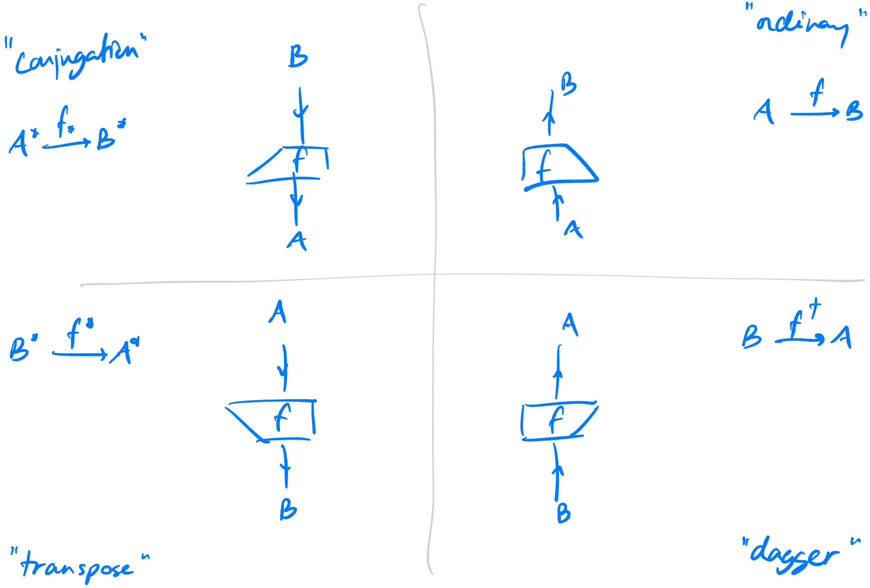
Draw conjugation by flipping the morphism about a vertical axis:



Since $(-)^*$ and $\dagger$ are contravariant, $(-)_*$ is covariant.


"conjugation"
A* f* B*
B
f
A
"ordinary"
A f B
B
f
A
B* f* A*
A
f
B
"transpose"
B f† A
A
f
B
"dagger"

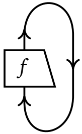
Conjugation: examples

Our examples **FHilb**, $\mathbf{Mat}_{\mathbb{C}}$ and **Rel** are all compact dagger categories

- In **FHilb**: conjugation functor gives conjugate of linear map
- In $\mathbf{Mat}_{\mathbb{C}}$: conjugation functor gives the conjugate of a matrix, each matrix entry replaced by its conjugate as a complex number
- In **Rel**: conjugation is identity

Trace and dimension

In a compact dagger category, the trace of a morphism $A \xrightarrow{f} A$ is the following scalar $\mathrm{Tr}_A(f)$:



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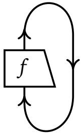

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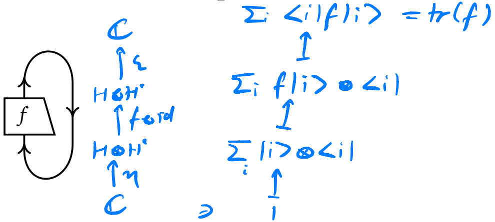

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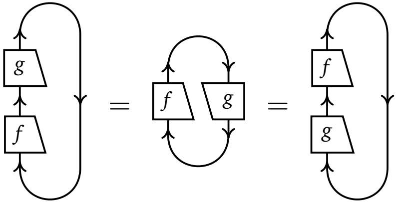
The trace in **FHilb** is the ordinary trace.

Trace is cyclic

In any compact dagger category, $\mathrm{Tr}_A(g \circ f) = \mathrm{Tr}_B(f \circ g)$.

Proof:

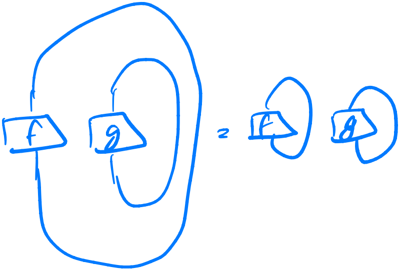


The g slides around the circle, and ends up underneath the f.

Trace and dimension properties

In a compact dagger category:

- $\mathrm{Tr}_I(s) = s$
- $\mathrm{Tr}_{A\otimes B}(f \otimes g) = \mathrm{Tr}_A(f) \circ \mathrm{Tr}_B(g)$
- $\left(\mathrm{Tr}_A(f)\right)^\dagger = \mathrm{Tr}_A(f^\dagger)$



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Hence:

- $\dim(I) = \mathrm{id}_I$
- $\dim(A \otimes B) = \dim(A) \circ \dim(B)$
- $A \simeq B \;\Rightarrow\; \dim(A) = \dim(B)$

Dual objects are finite-dimensional

Infinite-dimensional Hilbert spaces do not have duals.

Proof: Similarly we could prove $\dim(A \oplus B) = \dim(A) + \dim(B)$. Suppose H is an infinite-dimensional Hilbert space. Then there is an isomorphism $H \oplus \mathbb{C} \simeq H$. If H had a dual, then $\dim(H) + 1 = \dim(H)$. But this is a contradiction, since there is no complex number with that property.

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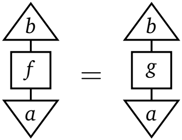
This argument would not apply in **Rel**, since there $\mathrm{id}_1 + \mathrm{id}_1 = \mathrm{id}_1$. Indeed, any set has a dual in **Rel**, even infinite ones.

Information flow

In well-pointed monoidal dagger category $f = g\colon A \to B$ if and only if



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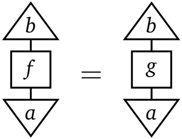
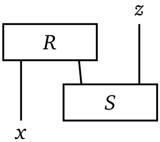
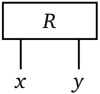


for all $a, b : I \to B$: can compare 'matrix entries'

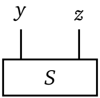
In **Rel** can conveniently decorate wires with elements: scalar



is 1 if and only if there is y such that following scalars both 1:



and



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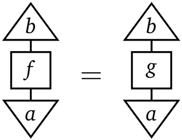

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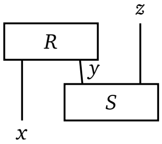
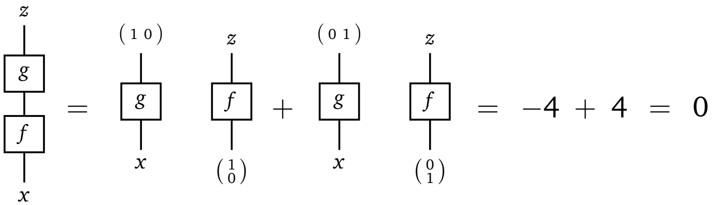
So can decorate



to signify that if x is connected to z, then must ‘flow’ through some y

Interference

In **FHilb**, have (destructive) interference:
if $g = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, $f = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $x = z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then



but both histories in the sum are possible

Cups are entangled

If $L \dashv R$, and $I \xrightarrow{\eta} R \otimes L$ is a product state,
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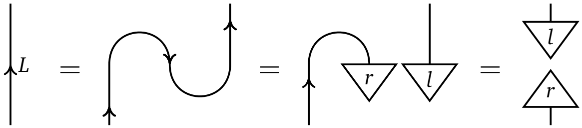
Proof: Suppose η is $I \xrightarrow{\lambda_i^{-1}} I \otimes I \xrightarrow{r \otimes l} R \otimes L$. Then:



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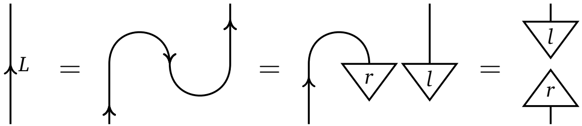

Interpreting diagram as history of events, disconnect means
output independent of input: L degenerate

Summary

- Dual objects: bend wires, maximally entangled states
- Names and conames: encode morphisms as states
- Dual morphisms: sliding, functorial
- Teleportation: quantum, one-time pad
- Graphical calculus for compact dagger categories: orientation
- Conjugation: combine duals with dagger
- Trace and dimension: turn morphisms into scalars
- Information flow: entanglement vs disconnect