Introduction to Quantum Programming and Semantics

Week 7: Frobenius structures

Chris Heunen
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Overview

- Frobenius structure: interacting co/monoid, self-duality
- Normal forms: coherence theorem
- Frobenius law: coherence between dagger and closure
- Classification: in $\mathbf{FHilb}$ and $\mathbf{Rel}$
- Phases: unitary operators
Idea

Orthonormal basis \( \{e_i\} \) for \( H \) in \( \text{FHilb} \) gives comonoid \( \varphi' : e_i \mapsto e_i \otimes e_i \). Its adjoint \( \Lambda \) is comparison: \( e_i \otimes e_i \mapsto e_i \) and \( e_i \otimes e_j \mapsto 0 \) if \( i \neq j \).

\[
\langle f(x) \mid y \rangle = \langle x \mid f^+(y) \rangle \\
\langle \varphi'(e_i) \mid e_j \otimes e_k \rangle = \langle e_i \mid \Lambda (e_j \otimes e_k) \rangle \\
\langle e_i \otimes e_i \mid e_j \otimes e_k \rangle \\
\langle e_i \mid e_j \rangle \langle e_i \mid e_k \rangle \\
\begin{cases} 
1 & \text{iff } i = j = k \\
0 & \text{otherwise}
\end{cases} \quad \text{or if } j \neq k \\
0 = \langle e_i \mid \Lambda (e_j \otimes e_k) \rangle \quad \forall i \\
0 = \varphi (e_j \otimes e_k) = 0
\]
Idea

Orthonormal basis \( \{e_i\} \) for \( H \) in \( \mathbf{FHilb} \) gives comonoid \( \varphi' : e_i \mapsto e_i \otimes e_i \). Its adjoint \( \triangleleft \) is comparison: \( e_i \otimes e_i \mapsto e_i \) and \( e_i \otimes e_j \mapsto 0 \) if \( i \neq j \).

These cooperate:

\[
\begin{array}{ccc}
\begin{array}{c}
\triangleleft \\
\downarrow \\
e_i \\
\end{array} & \quad & \begin{array}{c}
\triangleleft \\
\downarrow \\
e_j \\
\end{array} = \\
\begin{array}{c}
e_i \\
\downarrow \\
e_j \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
e_i \\
\downarrow \\
e_j \\
\end{array} & \quad & \begin{array}{c}
e_i \\
\downarrow \\
e_j \\
\end{array}
\end{array}
\]

This monoid/comonoid interaction is called the Frobenius law.
Idea

Orthonormal basis \( \{e_i\} \) for \( H \) in \textbf{FHilb} gives comonoid \( \wp: e_i \mapsto e_i \otimes e_i \). Its adjoint \( \wp' \) is \textit{comparison}: \( e_i \otimes e_i \mapsto e_i \) and \( e_i \otimes e_j \mapsto 0 \) if \( i \neq j \).

These cooperate:

\[
\begin{array}{ccc}
\begin{array}{c}
\ \\
\end{array} & \begin{array}{c}
\ \\
\end{array} & \begin{array}{c}
\ \\
\end{array} \\
\begin{array}{c}
e_i \\
\end{array} & \begin{array}{c}
e_j \\
\end{array} & \begin{array}{c}
e_j \\
\end{array}
\end{array}
= \begin{bmatrix}
e_i \\
\end{bmatrix} \begin{bmatrix}
e_j \\
\end{bmatrix}
\quad \text{if } i = j
\begin{bmatrix}
e_i \\
\end{bmatrix} \begin{bmatrix}0 \\
\end{bmatrix}
\quad \text{if } i \neq j
\end{array}
\]

This monoid/comonoid interaction is called the \textit{Frobenius law}. 
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A, \triangleright, \vartriangleleft)\) and monoid \((A, \triangleleft, \triangleright)\) satisfying the Frobenius law:

\[
\begin{align*}
\triangleright (\triangleright A, \vartriangleleft) = &
\triangleright A \triangleright (\vartriangleleft A, \vartriangleright) \\
= &
\vartriangleleft A \triangleright (\vartriangleleft A, \vartriangleright)
\end{align*}
\]

Examples of dagger Frobenius structures:

- In \(\text{FHilb}\): a Hilbert space equipped with an orthogonal basis.
- In \(\text{FHilb}\): let \(G\) be a finite group, spanning Hilbert space \(A\).
  - Define group algebra \(\mathbb{C}G\) with multiplication \(gh \mapsto gh\), and unit \(1 \mapsto 1_G\).
  - Adjoint: \(h \mapsto g(gh)\), and \(1 \mapsto g\).

- In \(\text{Rel}\): let \(G\) be a groupoid.
  - Monoid in \(\text{Rel}\): \((g, h) \mapsto gh\), and \((\cdot, \cdot) \mapsto \text{id}_X\).

- Frobenius law: \(gLh = \vartriangleleft gh\vartriangleleft\) for \(g = a \circ b\), \(t(h) = s(b)\).
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A, \triangleright, \triangleleft)\) and monoid \((A, \triangleleft, \bullet)\) satisfying the Frobenius law:

\[
\begin{array}{c}
\triangleleft \\
\downarrow \quad \downarrow \\
\triangleright \\
\end{array}
\quad =
\begin{array}{c}
\triangleright \\
\downarrow \quad \downarrow \\
\triangleleft \\
\end{array}
\]

If \(\triangleright = \triangleleft\), called dagger Frobenius structure.
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A, \Delta, \varepsilon)\) and monoid \((A, \cdot, 1)\) satisfying the Frobenius law:

\[
\begin{align*}
\Delta \Delta (a) &= \varepsilon (\varepsilon a) \\
\varepsilon \Delta (a) &= \Delta \varepsilon (a)
\end{align*}
\]

If \(\Delta = \varepsilon\), called dagger Frobenius structure.

Examples of dagger Frobenius structures:

- In \(\text{FHilb}\): a Hilbert space equipped with an orthogonal basis
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A, \triangleright, \bowtie)\) and monoid \((A, \triangleleft, \downarrow)\) satisfying the Frobenius law:

\[
\text{If } \triangleleft = \triangleright, \text{ called dagger Frobenius structure.}
\]

Examples of dagger Frobenius structures:

- In \(\text{FHilb}\): a Hilbert space equipped with an orthogonal basis
- In \(\text{FHilb}\): let \(G\) be finite group, spanning Hilbert space \(A\). Define group algebra \(\triangleleft\): \(g \otimes h \mapsto gh\), and \(\downarrow\): \(z \mapsto z \cdot 1_G\).
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A, \cdot, \bullet)\) and monoid \((A, \cdot, 1)\) satisfying the Frobenius law:

If \(\cdot = \cdot\), called dagger Frobenius structure.

Examples of dagger Frobenius structures:

- In \(\text{FHilb}\): a Hilbert space equipped with an orthogonal basis
- In \(\text{FHilb}\): let \(G\) be finite group, spanning Hilbert space \(A\).
  Define group algebra \(\cdot: g \otimes h \mapsto gh\), and \(\cdot: z \mapsto z \cdot 1_G\).
  Adjoint: \(\cdot: \sum_{h \in G} gh^{-1} \otimes h\), and \(\cdot: 1_G \mapsto g\) and \(1_G \neq g \mapsto 0\).

\(G = (\mathbb{Z}_3, +, 0)\)  \hspace{1cm} \(A = \mathbb{C}^3\)
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A,\circlearrowleft,\circlearrowright)\) and monoid \((A,\odot,\bullet)\) satisfying the Frobenius law:

If \(\circlearrowleft = \odot\), called dagger Frobenius structure.

Examples of dagger Frobenius structures:

- In \(\mathbf{FHilb}\): a Hilbert space equipped with an orthogonal basis
- In \(\mathbf{FHilb}\): let \(G\) be finite group, spanning Hilbert space \(A\).

Define group algebra \(\odot\): \(g \otimes h \mapsto gh\), and \(\bullet\): \(z \mapsto z \cdot 1_G\).

Adjoint: \(\circlearrowleft\): \(\sum_{h \in G} gh^{-1} \otimes h\), and \(\circlearrowright\): \(1_G \mapsto g\) and \(1_G \neq g \mapsto 0\).

Frobenius law: \(\text{LHS}(g \otimes h) = \sum_{k \in G} gk^{-1} \otimes kh = \text{RHS}(g \otimes h)\).

\[
\begin{align*}
L & = rh^{-1} \\
R & = kh
\end{align*}
\]
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A, \Delta, \epsilon)\) and monoid \((A, \cdot, 1)\) satisfying the Frobenius law:

\[
\Delta \cdot \epsilon = \epsilon \cdot \Delta
\]

If \(\Delta = \Delta\), called dagger Frobenius structure.

Examples of dagger Frobenius structures:

- In \(\mathbf{FHilb}\): a Hilbert space equipped with an orthogonal basis
- In \(\mathbf{FHilb}\): let \(G\) be finite group, spanning Hilbert space \(A\).
  Define group algebra \(\Delta: g \otimes h \mapsto gh\), and \(\epsilon: z \mapsto z \cdot 1_G\).
  Adjoint: \(\Delta: \sum_{h \in G} gh^{-1} \otimes h\), and \(\epsilon: 1_G \mapsto g\) and \(1_G \neq g \mapsto 0\).
  Frobenius law: \(\text{LHS}(g \otimes h) = \sum_{k \in G} gk^{-1} \otimes kh = \text{RHS}(g \otimes h)\).
- In \(\mathbf{Rel}\): let \(G\) be groupoid.
  Monoid in \(\mathbf{Rel}\): \(\Delta: (g, h) \sim g \circ h\), and \(\epsilon: \bullet \sim \text{id}_X\).
Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid \((A, \triangleright, \smile)\) and monoid \((A, \triangleleft, \bullet)\) satisfying the Frobenius law:

\[
\begin{array}{c}
\triangleright \smile \triangleleft = \\
\end{array}
\]

If \(\triangleleft = \triangleright\), called dagger Frobenius structure.

Examples of dagger Frobenius structures:

- In \(\text{FHilb}\): a Hilbert space equipped with an orthogonal basis.
- In \(\text{FHilb}\): let \(G\) be finite group, spanning Hilbert space \(A\). Define group algebra \(\triangleright\): \(g \otimes h \mapsto gh\), and \(\bullet\): \(z \mapsto z \cdot 1_G\).
  Adjoint: \(\triangleright\): \(\sum_{h \in G} gh^{-1} \otimes h\), and \(\bullet\): \(1_G \mapsto g\) and \(1_G \neq g \mapsto 0\).
  Frobenius law: \(\text{LHS}(g \otimes h) = \sum_{k \in G} g k^{-1} \otimes kh = \text{RHS}(g \otimes h)\).
- In \(\text{Rel}\): let \(G\) be groupoid.
  Monoid in \(\text{Rel}\): \(\triangleleft\): \((g, h) \sim g \circ h\), and \(\bullet\): \(\bullet \sim \text{id}_X\).
  Frobenius law: \((g, h) \sim (a, b \circ h)\) for \(g = a \circ b\), \(t(h) = s(b)\).
Pair of pants

In a dagger monoidal category, if $A \rightarrow A^*$, the pair of pants monoid $A^* \otimes A$ carries a dagger Frobenius structure.
Pair of pants

In a dagger monoidal category, if $A \rightarrow A^*$, the pair of pants monoid $A^* \otimes A$ carries a dagger Frobenius structure.

Proof.
Extended Frobenius law

Any Frobenius structure satisfies:

\[
\begin{array}{c}
\quad = \\
\quad = \\
\end{array}
\]
Extended Frobenius law

Any Frobenius structure satisfies:

\[
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad = \quad
\begin{tikzpicture}
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
\end{tikzpicture}
\]

Proof.

\[
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad = \quad
\begin{tikzpicture}
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
\end{tikzpicture}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad = \quad
\begin{tikzpicture}
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
\end{tikzpicture}
\]
Speciality

If copies orthogonal basis \( \{ e_i \} \), can find (squared) norm of \( e_i \):

\[
\begin{align*}
&e_i \\
&\downarrow \\
&e_i
\end{align*}
\]

and

\[
\begin{align*}
&e_i \\
&\downarrow \\
&e_i
\end{align*}
\]

\[
= 
\begin{align*}
&e_i \\
&\downarrow \\
&e_i
\end{align*}
\]

\[
\begin{align*}
&e_i \\
&\downarrow \\
&e_i
\end{align*}
\]

\[
\begin{align*}
&e_i \\
&\downarrow \\
&e_i
\end{align*}
\]

So can characterize orthonormality via Frobenius structure.

A Frobenius structure is special if:

Examples:

I Group algebra in \( \text{FHilb} \) is only special for trivial group

I Orthogonal basis in \( \text{FHilb} \) is special just when basis is orthonormal

I Groupoid Frobenius structure in \( \text{Rel} \) is always special
Speciality

If copies orthogonal basis \{e_i\}, can find (squared) norm of \(e_i\):

\[
\begin{align*}
\triangledown e_i \\
\downarrow e_i
\end{align*}
\]

and

\[
\begin{align*}
\triangledown e_i \\
\bigcirc \quad e_i \\
\bigcirc \quad e_i \\
\downarrow e_i \\
\downarrow e_i
\end{align*}
\]

\[
\begin{align*}
\triangledown e_i \\
\downarrow e_i \\
\triangledown e_i
\end{align*}
\]

So can characterize orthonormality via Frobenius structure.
Speciality

If \( \bigwedge \) copies orthogonal basis \( \{e_i\} \), can find (squared) norm of \( e_i \):

\[
\begin{align*}
\triangle e_i & \quad \text{and} \quad \circ \quad \triangle e_i \\
\triangle e_i & \quad \text{and} \quad \circ \quad \triangle e_i
\end{align*}
\]

So can characterize orthonormality via Frobenius structure.
A Frobenius structure is special if:

\[
\begin{align*}
\circ & = \quad \text{and} \quad \circ \\
\circ & = \quad \text{and} \quad \circ
\end{align*}
\]
Speciality

If copies orthogonal basis \( \{ e_i \} \), can find (squared) norm of \( e_i \):

\[
\begin{array}{c}
\begin{array}{c}
\triangle \quad e_i \\
\downarrow \\
\triangle \quad e_i \\
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
\blacklozenge \\
\end{array}
\end{array}
\end{array}
= \quad
\begin{array}{c}
\begin{array}{c}
\triangle \quad e_i \\
\downarrow \\
\triangle \quad e_i \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\triangle \quad e_i \\
\downarrow \\
\triangle \quad e_i \\
\end{array}
\end{array}

So can characterize orthonormality via Frobenius structure.

A Frobenius structure is special if:

\[
\begin{array}{c}
\begin{array}{c}
\blacklozenge \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\blacklozenge \\
\end{array}
\end{array}
\]

Examples:
Speciality

If copies orthogonal basis \( \{ e_i \} \), can find (squared) norm of \( e_i \):

\[ e_i e_i \]

and

\[ e_i = e_i \]

So can characterize orthonormality via Frobenius structure.

A Frobenius structure is special if:

Examples:
- Group algebra in \( FHilb \) is only special for trivial group
Speciality

If copies orthogonal basis \( \{ e_i \} \), can find (squared) norm of \( e_i \):

\[
\begin{align*}
\triangle e_i & \quad \triangle e_i \\
\triangle e_i & \quad \triangle e_i
\end{align*}
\]

and

\[
\begin{align*}
\bigcirc e_i & \quad \bigcirc e_i \\
\bigcirc e_i & \quad \bigcirc e_i
\end{align*}
\]

So can characterize orthonormality via Frobenius structure.

A Frobenius structure is special if:

\[
\begin{align*}
\bigcirc & \quad = \\
\bigcirc & \quad =
\end{align*}
\]

Examples:

- Group algebra in \( \text{FHilb} \) is only special for trivial group
- Orthogonal basis in \( \text{FHilb} \) is special just when basis is orthonormal
Speciality

If copies orthogonal basis \( \{e_i\} \), can find (squared) norm of \( e_i \):

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\end{array} &= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

and

So can characterize orthonormality via Frobenius structure.

A Frobenius structure is **special** if:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

Examples:

- Group algebra in **FHilb** is only special for trivial group
- Orthogonal basis in **FHilb** is special just when basis is orthonormal
- Groupoid Frobenius structure in **Rel** is always special
Classical structures
In a braided monoidal dagger category, a classical structure is a special commutative dagger Frobenius structure.
Classical structures

In a braided monoidal dagger category, a classical structure is a special commutative dagger Frobenius structure.

Examples:

- In $\text{FHilb}$: an orthonormal basis
- In $\text{Rel}$: abelian group
Classical structures

In a braided monoidal dagger category, a **classical structure** is a special commutative dagger Frobenius structure.

Examples:

- In **FHilb**: an orthonormal basis
- In **Rel**: abelian group

Definition of classical structure redundant:

- (Co)commutativity implies half of (co)unitality
- Speciality and Frobenius law imply (co)associativity
- Dual object and Frobenius law imply (co)unitality
Classical structures

In a braided monoidal dagger category, a classical structure is a special commutative dagger Frobenius structure.

Examples:
- In $\text{FHilb}$: an orthonormal basis
- In $\text{Rel}$: abelian group

Definition of classical structure redundant:
- (Co)commutativity implies half of (co)unitality
- Speciality and Frobenius law imply (co)associativity
- Dual object and Frobenius law imply (co)unitality

To check that $(A, \otimes, \Phi)$ is classical structure, only need:
Symmetry

Pair of pants hardly ever commutative. However:
A Frobenius structure is symmetric when:

\[
\begin{align*}
\text{Pair of pants: in } \text{FHilb} \text{ this says } & \quad \text{Tr}(ab) = \text{Tr}(ba) \\
\text{Group algebras: inverses in groups are two-sided inverses} & \\
\text{Groupoid Frobenius structure: inverses are two-sided} &
\end{align*}
\]
Symmetry

Pair of pants hardly ever commutative. However:
A Frobenius structure is symmetric when:

In a compact category, this is equivalent to the following:
Symmetry

Pair of pants hardly ever commutative. However:
A Frobenius structure is symmetric when:

\[
\begin{array}{c}
\quad =
\end{array}
\]

In a compact category, this is equivalent to the following:

\[
\begin{array}{c}
\quad =
\end{array}
\]

Examples:

- Pair of pants: in \textbf{FHilb} this says \( \text{Tr}(ab) = \text{Tr}(ba) \)
- Group algebras: inverses in groups are two-sided inverses
- Groupoid Frobenius structure: inverses are two-sided
Self-duality

If \((A, \Upsilon, \varphi, \otimes, \oplus)\) Frobenius structure in monoidal category, then \(A \rightarrow A\) is self-dual with:

\[
A \quad A \quad = \quad A \quad A
\]

\[
A \quad A \quad = \quad \bar{A}
\]
Self-duality

If \((A, \otimes, \varphi, \Delta, \epsilon)\) Frobenius structure in monoidal category, then \(A \rightarrow A\) is self-dual with:

\[
\begin{align*}
A & \quad A \\
\begin{array}{c}
\begin{array}{c}
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\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
= 
\begin{align*}
A & \quad A \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
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\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
A & \quad A \\
\begin{array}{c}
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\end{align*}
= 
\begin{align*}
A & \quad A \\
\begin{array}{c}
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\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
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\end{array}
\end{array}
\end{align*}
\]

Proof.

\[
\begin{align*}
A & \quad A \\
\begin{array}{c}
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\end{align*}
= 
\begin{align*}
A & \quad A \\
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\end{align*}
\]

\[
\begin{align*}
A & \quad A \\
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\end{align*}
= 
\begin{align*}
A & \quad A \\
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\begin{array}{c}
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\end{array}
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\end{array}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
A & \quad A \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array}
\end{array}
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\end{align*}
= 
\begin{align*}
A & \quad A \\
\begin{array}{c}
\begin{array}{c}
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\begin{array}{c}
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\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
A & \quad A \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
= 
\begin{align*}
A & \quad A \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]
Nondegenerate forms

Monoid \((A, \bullet, \dagger)\) forms Frobenius structure with comonoid \((A, \mathcal{F}, \phi)\) iff allows nondegenerate form: map \(\phi: A \to I\) with part of self-duality \(A \dashv A\).
Nondegenerate forms

Monoid \((A,\ast,\bullet)\) forms Frobenius structure with comonoid \((A,\forall,\forall)\) iff allows nondegenerate form: map \(\varphi: A \to I\) with

\[
\begin{array}{c}
\text{part of self-duality } A \dashv A.
\end{array}
\]

**Proof.** One direction is the previous theorem.
Nondegenerate forms

Monoid \((A, \bullet, \cdot)\) forms Frobenius structure with comonoid \((A, \mathcal{V}, \phi)\) iff allows nondegenerate form: map \(\phi: A \rightarrow I\) with

part of self-duality \(A \rightarrow A\).

**Proof.** One direction is the previous theorem. Conversely, suppose \(I \xrightarrow{\eta} A \otimes A\) satisfies:

\[
\begin{array}{c}
\xymatrix{
\circ \ar@{=>}[r] & I \\
A \ar[u] & \circ \\
\eta \ar[u] & \end{array}
\]

=  

\[
\begin{array}{c}
\xymatrix{
\circ \ar@{=>}[r] & I \\
A \ar[u] & \circ \\
\eta \ar[u] & \end{array}
\]

=  

\[
\begin{array}{c}
\xymatrix{
\circ \ar@{=>}[r] & I \\
\eta \ar[u] & I \\
\eta \ar[u] & \end{array}
\]
Nondegenerate forms

Monoid \((A, \bullet, \cdot)\) forms Frobenius structure with comonoid \((A, \mathcal{S}, \varphi)\) iff allows nondegenerate form: map \(\varphi: A \to I\) with

![Diagram]

part of self-duality \(A \cong A\).

**Proof.** One direction is the previous theorem. Conversely, suppose \(I \xrightarrow{\eta} A \otimes A\) satisfies:

\[
\begin{align*}
\eta &= \eta \\
\end{align*}
\]

Define comultiplication as:

\[
\begin{align*}
\eta &= \eta \\
\end{align*}
\]
Nondegenerate forms

**Proof** (continued.)

Could have defined the comultiplication with $\eta$ left or right:

\[
\begin{align*}
\eta & = \\
\eta & = \\
\eta & = \\
\eta & = \\
\end{align*}
\]
Nondegenerate forms

Proof (continued.)
Could have defined the comultiplication with $\eta$ left or right:

\[
\begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
= \begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
= \begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
= \begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
\end{array}
\]

(\ast)

Counitality:

\[
\begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
= \begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
= \begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
= \begin{array}{c}
\eta \\
\end{array}
\begin{array}{c}
\eta \\
\end{array}
\end{array}
\]

(\ast)
Nondegenerate forms

**Proof** (continued.)

Coassociativity:

\[
\eta \otimes \eta = \eta = \eta \otimes \eta = \eta \otimes \eta = \eta
\]
Nondegenerate forms

Proof (continued.)

Coassociativity:

Frobenius law:
Homomorphisms

A homomorphism of Frobenius structures is a morphism which is both monoid and comonoid homomorphism.
Homomorphisms

A homomorphism of Frobenius structures is a morphism which is both monoid and comonoid homomorphism. They are isomorphisms.

**Proof.** Given homomorphism $A \xrightarrow{f} B$, construct inverse as:

\[ f^{-1} = \]

![Diagram](image)
Homomorphisms

A homomorphism of Frobenius structures is morphism which is both monoid and comonoid homomorphism. They are isomorphisms.

**Proof.** Given homomorphism $A \xrightarrow{f} B$, construct inverse as:

$$f^{-1} = \text{diagram}$$

Indeed:

$$\text{diagram} = \text{diagram} = \text{diagram} = \text{diagram}$$
Normal forms

Two ways to think about graphical calculus:

- diagram represents morphism:
  merely shorthand to write down e.g. linear map;
- diagram is entity in its own right:
  can be manipulated by replacing equal parts.
Two ways to think about graphical calculus:

- diagram represents morphism: merely shorthand to write down e.g. linear map;
- diagram is entity in its own right: can be manipulated by replacing equal parts.

First viewpoint: ok if different diagrams represent same morphism.
Second viewpoint: combinatorial/graph theoretic flavour.
Normal forms

Two ways to think about graphical calculus:

- diagram represents morphism:
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First viewpoint: ok if different diagrams represent same morphism.
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A normal form theorem connects the two:
proving that all diagrams representing fixed morphism can be rewritten into canonical diagram (like coherence theorem)
Normal forms

Two ways to think about graphical calculus:
- diagram represents morphism:
  merely shorthand to write down e.g. linear map;
- diagram is entity in its own right:
  can be manipulated by replacing equal parts.

First viewpoint: ok if different diagrams represent same morphism.
Second viewpoint: combinatorial/graph theoretic flavour.
A normal form theorem connects the two:
proving that all diagrams representing fixed morphism can be rewritten into canonical diagram (like coherence theorem)
Unique way to copy (\(\triangleright\)), discard (\(\varnothing\)), fuse (\(\bowtie\)), create (\(\circ\)) data!
Let $(A, \Lambda, \odot, \otimes, \circ, \varphi)$ be a special Frobenius structure. Any connected morphism $A^\otimes m \to A^\otimes n$ built out of finitely many pieces $\Lambda, \odot, \otimes, \circ, \varphi,$ and $\text{id},$ using $\circ$ and $\otimes,$ equals:
Spider theorem

Let \((A, \bullet, \circ, \ast, \varphi)\) be a special Frobenius structure. Any connected morphism \(A^{\otimes m} \to A^{\otimes n}\) built out of finitely many pieces \(\bullet, \circ, \ast, \varphi, \text{id}\), using \(\circ\) and \(\otimes\), equals:

Proof. Induction on the number of dots.
Proof. (continued.)

Base case. Trivial, as the diagram must be one of $\bullet, \circ, \triangledown, \varnothing$. 

Proof. (continued.)

Base case. Trivial, as the diagram must be one of $\bullet$, $\bigcirc$, $\bigtriangledown$, $\wp$.

Induction step. Assume all diagrams with at most $n$ dots can be brought in normal form, and consider a diagram with $n + 1$ dots.
**Proof.** (continued.)

*Base case.* Trivial, as the diagram must be one of $\bullet$, $\circ$, $\bigcirc$, $\varnothing$.

*Induction step.* Assume all diagrams with at most $n$ dots can be brought in normal form, and consider a diagram with $n + 1$ dots. Use naturality to write diagram in form with topmost dot.
**Proof.** (continued.)

*Base case.* Trivial, as the diagram must be one of $\spider$, $\top$, $\bigcirc$, $\varphi$.

*Induction step.* Assume all diagrams with at most $n$ dots can be brought in normal form, and consider a diagram with $n + 1$ dots. Use naturality to write diagram in form with topmost dot.

- Topmost dot is $\varphi$: use counitality to eliminate it.
**Proof.** (continued.)

*Base case.* Trivial, as the diagram must be one of $\ast$, $\down$, $\Uparrow$, $\Downarrow$.

*Induction step.* Assume all diagrams with at most $n$ dots can be brought in normal form, and consider a diagram with $n + 1$ dots. Use naturality to write diagram in form with topmost dot.

- Topmost dot is $\ast$: use counitality to eliminate it.
- Topmost dot is $\down$: use coassociativity to reach normal form.
- Topmost dot is $\Uparrow$: the most interesting case. Is the diagram underneath connected? If so, use coassociativity and speciality.
Proof. (continued.)

Base case. Trivial, as the diagram must be one of \( \bullet, \cdot, \bigodot, \varphi \).

Induction step. Assume all diagrams with at most \( n \) dots can be brought in normal form, and consider a diagram with \( n + 1 \) dots. Use naturality to write diagram in form with topmost dot.

- Topmost dot is \( \varphi \): use counitality to eliminate it.
- Topmost dot is \( \bigodot \): impossible by connectedness.
Spider theorem

Proof. (continued.)

Base case. Trivial, as the diagram must be one of \(\bullet, \circ, \nabla, \varphi\).

Induction step. Assume all diagrams with at most \(n\) dots can be brought in normal form, and consider a diagram with \(n+1\) dots. Use naturality to write diagram in form with topmost dot.

1. Topmost dot is \(\varphi\): use counitality to eliminate it.
2. Topmost dot is \(\nabla\): use coassociativity to reach normal form.
3. Topmost dot is \(\circ\): impossible by connectedness.
4. Topmost dot is \(\bullet\): the most interesting case.

\[\text{Diagrams} \quad \Rightarrow \quad = \quad =\]
Proof. (continued.)

Base case. Trivial, as the diagram must be one of $\triangle$, $\overset{\triangledown}{\bullet}$, $\bigtriangleup$, $\bigtriangledown$.

Induction step. Assume all diagrams with at most $n$ dots can be brought in normal form, and consider a diagram with $n + 1$ dots. Use naturality to write diagram in form with topmost dot.

- Topmost dot is $\varphi$: use counitality to eliminate it.
- Topmost dot is $\bigtriangleup$: use coassociativity to reach normal form.
- Topmost dot is $\overset{\triangledown}{\bullet}$: impossible by connectedness.
- Topmost dot is $\triangle$: the most interesting case.

Is the diagram underneath the $\triangle$ connected? If so, use coassociativity and speciality.
Spider theorem

Proof. (continued.)

Suppose instead the rest of the diagram is disconnected:
More spider theorems

In a monoidal category, let \((A, \triangleright, \triangleleft, \gamma, \varphi)\) be a Frobenius structure. Any connected morphism \(A^\otimes m \to A^\otimes n\) built out of finitely many pieces \(\triangleright, \triangleleft, \gamma, \varphi,\) and \(\text{id},\) using \(\circ\) and \(\otimes,\) equals \((\ast)\).

\[
\ast
\]
More spider theorems

In a monoidal category, let \((A, \otimes, \cdot, \varphi, \varphi')\) be a Frobenius structure. Any connected morphism \(A^\otimes m \to A^\otimes n\) built out of finitely many pieces, \(\otimes, \cdot, \varphi, \varphi', \text{id}\), using \(\circ\) and \(\otimes\), equals (⋆).

In a symmetric monoidal category, let \((A, \otimes, \cdot, \varphi, \varphi')\) be a commutative Frobenius structure. Any connected morphism \(A^\otimes m \to A^\otimes n\) built out of finitely many pieces, \(\otimes, \cdot, \varphi, \varphi', \text{id}, \varphi\), using \(\circ\) and \(\otimes\), equals (⋆).
No braided spider theorem

In a braided non-symmetric monoidal category, there is no normal form for commutative Frobenius algebras.
No braided spider theorem

In a braided non-symmetric monoidal category, there is no normal form for commutative Frobenius algebras.

**Proof.** Regard the following diagram as a piece of string on which an overhand knot is tied:
No braided spider theorem

In a braided non-symmetric monoidal category, there is no normal form for commutative Frobenius algebras.

**Proof.** Regard the following diagram as a piece of string on which an overhand knot is tied:

The Frobenius algebra axioms induce homotopy equivalences (‘deformations’) of the corresponding graph. Such moves cannot untie the knot.
Involutive monoids

If \((A, m, u)\) is monoid, so is \((A^*, m^*, u^*)\).
Involution monoids

If \((A, m, u)\) is monoid, so is \((A^*, m_*, u_*)\).

An **involution** for a monoid \((A, \odot, \odot)\) is a monoid homomorphism \(A \xrightarrow{i} A^*\) satisfying \(i_* \circ i = \text{id}_A\).
Involution monoids

If $(A, m, u)$ is monoid, so is $(A^*, m^*_*, u^*_*)$.

An **involution** for a monoid $(A, \cdot, \circ)$ is a monoid homomorphism $A \xrightarrow{i} A^*$ satisfying $i_* \circ i = \text{id}_A$.

A **morphism of involutive monoids** is monoid homomorphism $A \xrightarrow{f} B$ satisfying $i_B \circ f = f_* \circ i_A$. 
Example involutive monoids

- Matrix algebra. $\mathbb{M}_n$ is an involutive monoid in $\text{FHilb}$. 
  Opposite monoid $\mathbb{M}_n^*$: multiplication $ab$ in $\mathbb{M}_n^*$ is $ba$ in $\mathbb{M}_n$. 
  Canonical involution $\mathbb{M}_n \to \mathbb{M}_n^*$ given by $f \mapsto f^\dagger$. 

Pair of pants. $A^\dagger \mapsto A$ involutive in a dagger pivotal category. 
Identity map as involution, because of conventions: $\text{Identity map as involution, because of conventions:}$. 

Groupoids. $G$ in $\text{Rel}$ is involutive. 
Opposite monoid: induced by opposite groupoid $G^\text{op}$. 
Canonical involution $G \to G^\dagger$ given by $g \mapsto g^{-1}$. 

Example involutive monoids

- **Matrix algebra.** $\mathbb{M}_n$ is an involutive monoid in $\text{FHilb}$. Opposite monoid $\mathbb{M}_n^*$: multiplication $ab$ in $\mathbb{M}_n^*$ is $ba$ in $\mathbb{M}_n$. Canonical involution $\mathbb{M}_n \to \mathbb{M}_n^*$ given by $f \mapsto f^\dagger$.

- **Pair of pants.** $A^* \otimes A$ involutive in a dagger pivotal category. Identity map as involution, because of conventions:

\[
\begin{array}{c}
(\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array})^* = \left(\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}\right)^\dagger = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
\end{array}
\]
Example involutive monoids

- Matrix algebra. $\mathbb{M}_n$ is an involutive monoid in $\mathbf{FHilb}$. Opposite monoid $\mathbb{M}_n^*$: multiplication $ab$ in $\mathbb{M}_n^*$ is $ba$ in $\mathbb{M}_n$. Canonical involution $\mathbb{M}_n \rightarrow \mathbb{M}_n^*$ given by $f \mapsto f^\dagger$.

- Pair of pants. $A^* \otimes A$ involutive in a dagger pivotal category. Identity map as involution, because of conventions:

\[
\left( \begin{array}{c}
\end{array} \right)_* = \left( \begin{array}{c}
\end{array} \right)^\dagger = \left( \begin{array}{c}
\end{array} \right)
\]

- Groupoids. $G$ in $\mathbf{Rel}$ is involutive. Opposite monoid: induced by opposite groupoid $G^{\text{op}}$

\[
\left( \begin{array}{c}
\end{array} \right) = \left( \begin{array}{c}
\end{array} \right)
\]

Canonical involution $G \rightarrow G^*$ given by $g \sim g^{-1}$. 

Frobenius law from way of the dagger

Monoid \((A, \cdot, e)\) is dagger Frobenius if and only if \(i\) is involution:

\[
\begin{array}{c}
\text{i} \\
\downarrow
\end{array}
= 
\begin{array}{c}
\text{\_} \\
\text{\_}
\end{array}
\]
Frobenius law from way of the dagger

Monoid \((A, \cdot, \mathcal{O})\) is dagger Frobenius if and only if \(i\) is involution:

\[
\begin{array}{c}
\text{i} \\
\downarrow
\end{array} = \begin{array}{c}
\text{=}
\end{array}
\]

Proof. Assume dagger Frobenius.
Frobenius law from way of the dagger

Monoid \((A, \wedge, \dagger)\) is dagger Frobenius if and only if \(i\) is involution:

\[
\begin{array}{cccc}
 & & & \\
& & & \\
i & = & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

**Proof.** Assume dagger Frobenius.

- \(i\) preserves multiplication:
Frobenius law from way of the dagger

Monoid \((A, \cdot, \mathbf{1})\) is dagger Frobenius if and only if \(i\) is involution:

\[
i = \quad = \quad = \quad = \quad = \quad = \quad \]

\[
\begin{array}{c}
i \\
\downarrow \\
\end{array} 
\]

\[
\begin{array}{c}
i \\
\downarrow \\
\end{array} 
\]

**Proof.** Assume dagger Frobenius.

- \(i\) preserves multiplication:

\[
\begin{array}{c}
i \\
\downarrow \\
\end{array} 
\quad = \quad = \quad = \quad = \quad \begin{array}{c}
i \\
\downarrow \\
\end{array} 
\]

- \(i\) preserves units: easy.
Frobenius law from way of the dagger

Monoid \((A, \cdot, 1)\) is dagger Frobenius if and only if \(i\) is involution:

\[
\begin{array}{c}
\begin{array}{c}
\text{Proof. Assume dagger Frobenius.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{\(i\) preserves multiplication:}
\end{array}
\]

\[
\begin{array}{c}
\text{\(i\) preserves units: easy.}
\end{array}
\]

\[
\begin{array}{c}
\text{\(i\) is involution:}
\end{array}
\]
Proof. (continued.) Conversely, suppose $i^* \circ i = \text{id}$. Then:

\[
\begin{align*}
\text{and by applying } ^\dagger, \quad \text{we have a Frobenius structure, defined by a nondegenerate form.}
\end{align*}
\]
Frobenius law from way of the dagger

**Proof.** (continued.) Conversely, suppose $i_* \circ i = \text{id}$. Then:

$$
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{proof1.png}
\end{array}
\end{array} &=& \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{proof2.png}
\end{array}
\end{array}
\end{array}
\end{array}
$$

and by applying $\dagger$, 

$$
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{proof3.png}
\end{array}
\end{array} &=& \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{proof4.png}
\end{array}
\end{array}
\end{array}
\end{array}
$$

So we have a Frobenius structure, defined by a nondegenerate form. Is it a dagger Frobenius structure?
Frobenius law from way of the dagger

**Proof.** (continued.) Conversely, suppose $i_* \circ i = \text{id}$. Then:

$$
\begin{align*}
\left(\begin{array}{c}
\end{array}\right) & = 1 \\
\left(\begin{array}{c}
\end{array}\right) & = 1
\end{align*}
$$

and by applying $\dagger$, 

So we have a Frobenius structure, defined by a nondegenerate form. Is it a dagger Frobenius structure?

The condition that $i$ preserves multiplication gives:

$$
\left(\begin{array}{c}
\end{array}\right) \Rightarrow \left(\begin{array}{c}
\end{array}\right) \Rightarrow \left(\begin{array}{c}
\end{array}\right)
$$

So the form definition gives rise to the correct comultiplication.
Classification in FHilb

**Theorem:** special dagger Frobenius structures in FHilb are of the form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$.
Theorem: special dagger Frobenius structures in $\text{FHilb}$ are of the form $\mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_k}$.

Proof:

- Cayley: dagger Frobenius structure on $H$ embeds into $H^* \otimes H$
- $H^* \otimes H$ isomorphic to $\mathbb{M}_{\dim(H)}$
- so $H$ involutive subalgebra of $\mathbb{M}_{\dim(H)}$: C*-algebra
- Artin-Wedderburn: must be of form $\mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_k}$
Classification in $\text{FHilb}$

**Theorem:** special dagger Frobenius structures in $\text{FHilb}$ are of the form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$.

**Proof:**

- **Cayley:** dagger Frobenius structure on $H$ embeds into $H^* \otimes H$
- $H^* \otimes H$ isomorphic to $\mathbb{M}_{\dim(H)}$
- so $H$ involutive subalgebra of $\mathbb{M}_{\dim(H)}$: C$^*$-algebra
- **Artin-Wedderburn:** must be of form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$

**Corollary:** classical structure in $\text{FHilb}$ copy orthonormal bases

**Proof:** must be of form $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$. 
Orthogonal bases

Frobenius structure that copies basis is dagger Frobenius if and only if basis is orthogonal.
Orthogonal bases

Frobenius structure that copies basis is dagger Frobenius if and only if basis is orthogonal.

**Proof.** For nonzero copyable states:

\[
\begin{align*}
\triangle x & \triangle x \triangle y |
\triangle x \triangle x \triangle x & = \\
\triangle x & \triangle y |
\triangle x \triangle x \triangle y & = \\
\triangle x \triangle y & \triangle x \triangle x \triangle y & = \\
\triangle x \triangle x \triangle x & \triangle x \triangle x \triangle x & = 
\end{align*}
\]

If \( h_x | y \rangle = 0 \), then this is satisfied.

If \( h_x | y \rangle \neq 0 \), this implies \( h_x | x \rangle = h_x | y \rangle \).

Similarly \( h_y | x \rangle = h_y | y \rangle \).

Hence \( h_x y | x y \rangle = h_x | x \rangle h_x | y \rangle + h_y | y \rangle = 0 \), so \( x = y \).
Orthogonal bases

Frobenius structure that copies basis is dagger Frobenius if and only if basis is orthogonal.

**Proof.** For nonzero copyable states:

If $\langle x|y \rangle = 0$, then this is satisfied.
Orthogonal bases

Frobenius structure that copies basis is dagger Frobenius if and only if basis is orthogonal.

**Proof.** For nonzero copyable states:

If $\langle x|y \rangle = 0$, then this is satisfied.

If $\langle x|y \rangle \neq 0$, this implies $\langle x|x \rangle = \langle x|y \rangle$. Similarly $\langle y|x \rangle = \langle y|y \rangle$. 
Orthogonal bases

Frobenius structure that copies basis is dagger Frobenius if and only if basis is orthogonal.

**Proof.** For nonzero copyable states:

\[
\begin{align*}
\langle x | y \rangle &= \langle x | x \rangle \\
\langle y | x \rangle &= \langle y | y \rangle
\end{align*}
\]

If \( \langle x | y \rangle = 0 \), then this is satisfied.

If \( \langle x | y \rangle \neq 0 \), this implies \( \langle x | x \rangle = \langle x | y \rangle \). Similarly \( \langle y | x \rangle = \langle y | y \rangle \).

Hence \( \langle x - y | x - y \rangle = \langle x | x \rangle - \langle x | y \rangle - \langle y | x \rangle + \langle y | y \rangle = 0 \), so \( x = y \).
Orthogonal bases and morphisms

In $\text{FHilb}$, morphism between two commutative dagger Frobenius structures acts as function on copyable states if and only if it is comonoid homomorphism.
Orthogonal bases and morphisms

In $\mathbf{FHilb}$, morphism between two commutative dagger Frobenius structures acts as function on copyable states if and only if it is comonoid homomorphism.

**Proof.** Suffices to see about basis of copyable states $\{e_i\}$.

Hence $f(e_i)$ copyable.
Classification in $\textbf{Rel}$

**Theorem:** Special dagger Frobenius structures in $\textbf{Rel}$ correspond exactly to groupoids.
Classification in \textbf{Rel}

**Theorem:** Special dagger Frobenius structures in \textbf{Rel} correspond exactly to groupoids.

**Proof.** Write $A \times A \xrightarrow{M} A$ for multiplication, $U \subseteq A$ for unit.
**Theorem:** Special dagger Frobenius structures in \textbf{Rel} correspond exactly to groupoids.

**Proof.** Write $A \times A \xrightarrow{M} A$ for multiplication, $U \subseteq A$ for unit.

$M$ is single-valued: by speciality $a(M \circ M^\dagger)b$ iff $a = b$:

$$
\begin{array}{c}
\quad

\end{array}
$$

So: if $(c, d)Ma$ and $(c, d)Mb$, must have $a = b$.

May simply write $ab$ for unique $c$ with $(a, b)Mc$.

Remember: $ab$ not always defined!
Classification in \textit{Rel}

\textbf{Proof.} (continued)

\textit{Associativity:}

So $ab$ and $(ab)c$ defined exactly when $bc$ and $a(bc)$ are defined, and then $(ab)c = a(bc)$. 
**Classification in Rel**

**Proof.** (continued)

*Unitality:* for units $x, y \in U$

So: $a, b$ allow $x \in U$ with $xa = b$ iff $a = b$.
And: $a, b$ allow $y \in U$ with $ay = b$ iff $a = b$.
If $z \in U$ then $xz = x$ for some $x \in U$. But then $x = z$!
Units idempotent; multiplication of different ones undefined.
If $xa = a = x'a$, then $a = xa = x(x'a) = (xx')a$, so $x = x'$.
So every element has unique left/right identity.
Classification in \textbf{Rel}

**Proof.** (continued)

*Category:* \(U \) set of objects, \(A \) set of morphisms.

If \(fg\) defined and \(gh\) defined, want \((fg)h = f(gh)\) defined too:

\[
\begin{array}{ccc}
fg & h \\
\uparrow & \downarrow \\
g & \quad \\
\downarrow & \uparrow \\
f & gh
\end{array}
\quad = \quad
\begin{array}{ccc}
fg & h \\
\uparrow & \downarrow \\
f(gh) = (fg)h \\
\downarrow & \uparrow \\
f & gh
\end{array}
\]

If \(fg\) and \(gh\) defined then LHS defined, so RHS defined too.
Classification in Rel

Proof. (continued)

Inverses: for \( f \in A \) with left unit \( x \) and right unit \( y \):

\[
x f = x y f = x f y
\]
Phases

Let \((A, \otimes, \delta)\) be Frobenius structure in a monoidal dagger category. State \(I \xrightarrow{a} A\) is called phase when:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
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Phases

Let \((A, \otimes, \Diamond)\) be Frobenius structure in a monoidal dagger category. State \(I \xrightarrow{a} A\) is called phase when:

\[
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\end{align*}
\]

Its (right) phase shift is the following morphism \(A \to A\):

\[
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& =
\begin{array}{c}
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\end{array}
& \begin{array}{c}
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\end{array}
\end{align*}
\]
Example phases

- For classical structure in $\textbf{FHilb}$ copying basis $\{e_i\}$, vector $a = a_1e_1 + \cdots a_ne_n$ is phase iff each $a_i$ on unit circle: $|a_i|^2 = 1$. 

The unit of a Frobenius structure is always a phase.

In a compact dagger category, phases for pair of pants $(A \xrightarrow{\cdot} I \xleftarrow{\cdot} A)$ correspond to unitary morphisms.

Proof.

The name of a morphism $A \xrightarrow{f} A$ is a phase when:

But this means $f \cdot f^\dagger = \text{id}_A$; similarly $f^\dagger \cdot f = \text{id}_A$. 

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Example phases

- For classical structure in $\text{FHilb}$ copying basis $\{e_i\}$, vector $a = a_1 e_1 + \cdots + a_n e_n$ is phase iff each $a_i$ on unit circle: $|a_i|^2 = 1$.

- The unit $\phi$ of a Frobenius structure is always a phase.
Example phases

- For classical structure in \textbf{FHilb} copying basis \{e_i\}, vector $a = a_1e_1 + \cdots + a_ne_n$ is phase iff each $a_i$ on unit circle: $|a_i|^2 = 1$.
- The unit $\phi$ of a Frobenius structure is always a phase.
- In a compact dagger category, phases for pair of pants $(A^* \otimes A, \langle, \rangle, \rho)$ correspond to unitary morphisms.

**Proof.**
Example phases

- For classical structure in \( \text{FHilb} \) copying basis \( \{e_i\} \), vector \( a = a_1 e_1 + \cdots + a_n e_n \) is phase iff each \( a_i \) on unit circle: \( |a_i|^2 = 1 \).
- The unit \( \mathfrak{O} \) of a Frobenius structure is always a phase.
- In a compact dagger category, phases for pair of pants \((A^* \otimes A, \langle, \rangle, \cup)\) correspond to unitary morphisms.

**Proof.** The name of an morphism \( A \xrightarrow{f} A \) is a phase when:

\[
\begin{array}{c}
\xymatrix{ & f \\
& f' \\
& f'' \\
& f'''
}\end{array}
\]

But this means \( f \circ f^\dagger = \text{id}_A \); similarly \( f^\dagger \circ f = \text{id}_A \).
Example phases

Phases of Frobenius structure $\mathbb{M}_n$ in $\text{FHilb}$ form set $U(n)$ of $n$-by-$n$ unitary matrices. Hence phases of $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ range over $U(k_1) \times \cdots \times U(k_n)$. 

Proof. For a subset $a \subseteq G$, equation defining phases reads

\[ \{ g_1 h \mid g, h \in a \} = \{ 1_G \} = \{ hg_1 \mid g, h \in a \}. \]

So if $g_2 \in G$, then $a = \{ g \}$ is a phase. But if $a$ contains distinct elements $g \neq h$ of $G$, cannot be phase. Similarly, $a = \emptyset$ not phase. Hence $a$ phase precisely when singleton $\{ g \}$. 

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Example phases

- Phases of Frobenius structure $\mathbb{M}_n$ in $\text{FHilb}$ form set $U(n)$ of $n$-by-$n$ unitary matrices. Hence phases of $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ range over $U(k_1) \times \cdots \times U(k_n)$.

- Classical structure $\mathbb{C}^n$ copying basis $\{e_1, \ldots, e_n\}$. Phases are elements of $U(1) \times \cdots \times U(1)$; phase shift $\mathbb{C}^n \to \mathbb{C}^n$ is accompanying unitary matrix.
Example phases

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- The phases of a Frobenius structure in $\text{Rel}$ induced by a group $G$ are elements of that group $G$ itself.
Example phases

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- The phases of a Frobenius structure in $\text{Rel}$ induced by a group $G$ are elements of that group $G$ itself.

Proof. For a subset $a \subseteq G$, equation defining phases reads

$$\{g^{-1}h \mid g, h \in a\} = \{1_G\} = \{hg^{-1} \mid g, h \in a\}.$$ 

So if $g \in G$, then $a = \{g\}$ is a phase. But if $a$ contains distinct elements $g \neq h$ of $G$, cannot be phase. Similarly, $a = \emptyset$ not phase. Hence $a$ phase precisely when singleton $\{g\}$. 

Phase group

In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit $\mathbf{1}$ and:

$$a + b = a \cdot b = b \cdot a = b + a$$

Proof. This is again a well-defined phase:

$$a + b = a \cdot b = b \cdot a = b + a$$

The flipped equation follows similarly.

Associativity is clear, hence phases form a monoid.
Phase group

In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit $\circ$ and:

\[
\begin{align*}
\quad & a + b \quad = \\
\quad & a \quad b
\end{align*}
\]

**Proof.** This is again a well-defined phase:

The flipped equation follows similarly.

Associativity is clear, hence phases form a monoid.
Phase group

Proof. (continued)

Left-inverse of phase $a$ is:

\[ -a = a \]

Similarly there is right-inverse. But in monoids, left and right inverses are equal:

\[ l = l \cdot x \cdot r = l \cdot x = r = r \cdot x \]
Phase group

**Proof.** (continued)

Left-inverse of phase \( a \) is:

\[
\left( a\right) + a = a
\]

Similarly there is right-inverse. But in monoids, left and right inverses are equal:

\[
l = l \cdot r = r = r.
\]
Phase group

Proof. (continued)

Left-inverse of phase $a$ is:

![Diagram showing $a$ and $a$]

Left-inverse of $a$ is $-a$:

![Diagram showing $-(a) + a$ and $a$]

Similarly there is right-inverse. But in monoids, left and right inverses are equal: $l = l(xr) = (lx)r = r$. 

\[\square\]
Example phase groups

- In \textbf{FHilb}, the phase group for the pair of pants Frobenius structure is the unitary group.
Example phase groups

- In **FHilb**, the phase group for the pair of pants Frobenius structure is the unitary group.

- Phase addition in the Frobenius structure $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ in **FHilb** is entrywise multiplication in $U(k_1) \times \cdots \times U(k_n)$. In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.
Example phase groups

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- Phase addition in the Frobenius structure $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ in **FHilb** is entrywise multiplication in $U(k_1) \times \cdots \times U(k_n)$. In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.

- In **Rel**, the phase group induced by a group $G$ is the group itself.
**Phased spider theorem**

Let \((A, \otimes, \delta)\) be classical structure in braided monoidal dagger category. Any connected morphism \(A^\otimes m \to A^\otimes n\) built of finitely many \(\otimes, \delta, \text{id}, \sigma\) and phases using \(\circ, \otimes, \text{ and } \dagger\), equals

![Diagram](image)

where \(a\) ranges over all the phases used in the diagram.
Phased spider theorem

Let \((A, \otimes, \odot)\) be classical structure in braided monoidal dagger category. Any connected morphism \(A^\otimes m \to A^\otimes n\) built of finitely many \(\otimes, \odot, \text{id}, \sigma\) and phases using \(\circ, \otimes, \) and \(\dagger\), equals

\[
\begin{array}{c}
\sum a \\
\end{array}
\]

where \(a\) ranges over all the phases used in the diagram.

**Proof.** Use braidings to have all phases dangle at the bottom. Apply Spider Theorem. Use phase addition to reduce to single phase \(\sum a\) on bottom right. Apply Spider Theorem again. \(\square\)
State transfer

State transfer protocol: transfer state of Hilbert space $H$ from one system to another, with success probability $1 / \dim(H)^2$.

May be lax in drawing, e.g. projection $H \otimes H \rightarrow H \otimes H$:

The procedure looks like this:

Extra challenge: apply phase gate while transferring state
Summary

- Frobenius structures: interacting co/monoid, self-duality
- Normal forms: spider theorems
- Frobenius law: justified by coherence
- Classification: matrix algebras, bases, groupoids
- Phases: unitary operators, state transfer

Next week: interaction between two Frobenius structures