

Introduction to Quantum Programming and Semantics

Week 7: Frobenius structures

Chris Heunen



THE UNIVERSITY *of* EDINBURGH
informatics

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Overview

- ▶ Frobenius structure: interacting co/monoid, self-duality
- ▶ Normal forms: coherence theorem
- ▶ Frobenius law: coherence between dagger and closure
- ▶ Classification: in **FHilb** and **Rel**
- ▶ Phases: unitary operators

Idea

Plancherel:

$$x = \sum_i \langle e_i | x \rangle e_i$$



Orthonormal basis $\{e_i\}$ for H in **FHilb** gives comonoid $\varphi: e_i \mapsto e_i \otimes e_i$.
 Its adjoint ρ is **comparison**: $e_i \otimes e_i \mapsto e_i$ and $e_i \otimes e_j \mapsto 0$ if $i \neq j$.

$$\langle f(x) | y \rangle = \langle x | f^\dagger(y) \rangle$$

$$\langle \varphi(e_i) | e_j \otimes e_k \rangle = \langle e_i | \rho(e_j \otimes e_k) \rangle$$

$$\langle e_i \otimes e_i | e_j \otimes e_k \rangle$$

$$\langle e_i | e_j \rangle \langle e_i | e_k \rangle$$

$$\begin{cases} 1 & \text{iff } i=j=k \\ 0 & \text{otherwise} \end{cases}$$

so if $j=k$ pick $i=j$ and:

$$1 = \langle e_i | \rho(e_i \otimes e_i) \rangle$$

$$\text{i.e. } e_i = \rho(e_i \otimes e_i)$$

or if $j \neq k$

$$0 = \langle e_i | \rho(e_j \otimes e_k) \rangle \quad \forall i$$

$$\text{so } \rho(e_j \otimes e_k) = 0$$

Idea

Orthonormal basis $\{e_i\}$ for H in **FHilb** gives comonoid $\complement: e_i \mapsto e_i \otimes e_i$.
Its adjoint \lrcorner is **comparison**: $e_i \otimes e_i \mapsto e_i$ and $e_i \otimes e_j \mapsto 0$ if $i \neq j$.

These cooperate:

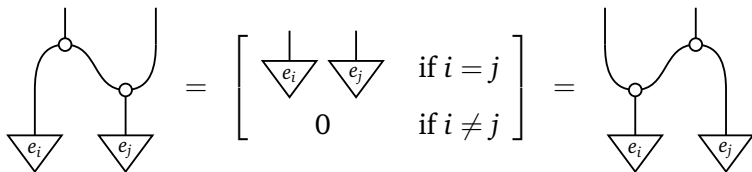
The diagrammatic equation illustrates the cooperation between comultiplication and comparison. On the left, a comultiplication node (a circle with two outgoing lines) is connected to two comparison nodes (triangles pointing down). The top line of the comultiplication node goes to the top of the left comparison node, which outputs e_i . The bottom line of the comultiplication node goes to the top of the right comparison node, which outputs e_j . This is equal to a piecewise definition: a matrix with two rows. The top row contains two comparison nodes (triangles pointing down) with outputs e_i and e_j , followed by the text "if $i = j$ ". The bottom row contains the text "0" followed by "if $i \neq j$ ". This is equal to a diagram where the comultiplication node is connected to two comparison nodes. The top line of the comultiplication node goes to the top of the left comparison node, which outputs e_i . The bottom line of the comultiplication node goes to the top of the right comparison node, which outputs e_j .

$$\begin{array}{c} \text{---} \\ | \\ \circ \\ | \quad \text{---} \\ \text{---} \quad \text{---} \\ | \quad | \\ \triangle \quad \triangle \\ e_i \quad e_j \end{array} = \begin{bmatrix} \begin{array}{c} \text{---} \\ | \\ \triangle \\ e_i \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \triangle \\ e_j \end{array} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{bmatrix} = \begin{array}{c} \text{---} \\ | \\ \circ \\ | \quad \text{---} \\ \text{---} \quad \text{---} \\ | \quad | \\ \triangle \quad \triangle \\ e_i \quad e_j \end{array}$$

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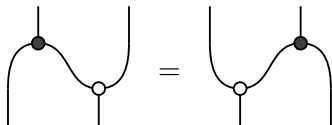
These cooperate:



This monoid/comonoid interaction is called the **Frobenius law**.

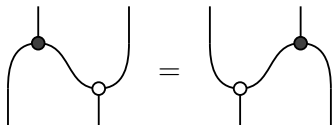
Frobenius structures

In a monoidal category, a **Frobenius structure** is a comonoid (A, ψ, φ) and monoid (A, μ, ν) satisfying the **Frobenius law**:



Frobenius structures

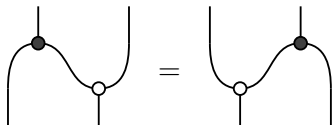
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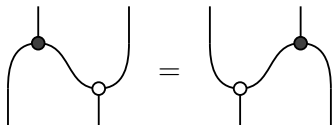
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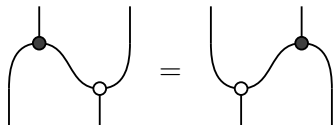
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Adjoint: $\psi: \sum_{h \in G} gh^{-1} \otimes h$, and $\varphi: 1_G \mapsto \sum_{g \in G} g$ and $1_G \neq g \mapsto 0$.

$$G = (\mathbb{Z}_3, +, 0)$$

$$A = \mathbb{C}^3$$

$$\begin{array}{ccc} \bullet & \mathbb{C}^3 & \longrightarrow \mathbb{C} \\ \downarrow & \parallel & \parallel \\ \bullet & \mathbb{C}^3 & \longrightarrow \mathbb{C} \\ \downarrow & \parallel & \parallel \\ \bullet & \mathbb{C}^3 & \longrightarrow \mathbb{C} \\ \downarrow & \parallel & \parallel \\ \bullet & \mathbb{C}^3 & \longrightarrow \mathbb{C} \end{array}$$

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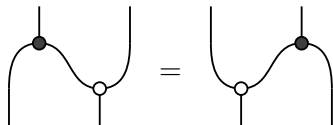
Frobenius law: $\text{LHS}(g \otimes h) = \sum_{k \in G} gk^{-1} \otimes kh = \text{RHS}(g \otimes h)$.

$$k = rh^{-1}$$

$$l = kh$$

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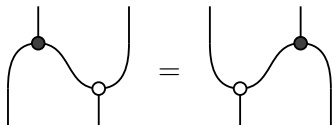
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- ▶ In **Rel**: let \mathbf{G} be **groupoid**.

Monoid in **Rel**: μ : $(g, h) \sim g \circ h$, and ν : $\bullet \sim \text{id}_X$.

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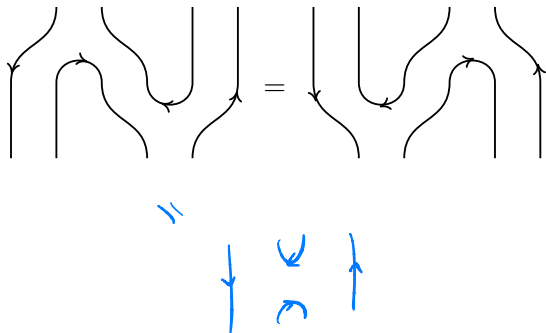
Pair of pants

In a dagger monoidal category, if $A \dashv A^*$, the pair of pants monoid $A^* \otimes A$ carries a dagger Frobenius structure.

Pair of pants

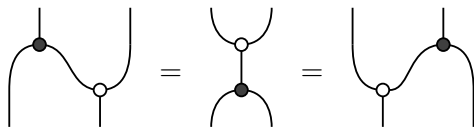
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Proof.



Extended Frobenius law

Any Frobenius structure satisfies:



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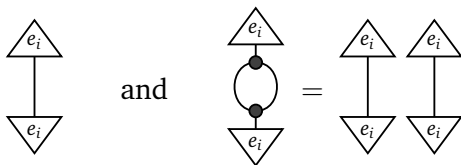
A diagrammatic equation showing the extended Frobenius law. It consists of three diagrams connected by equals signs. The first diagram shows a black dot on top and a white dot on bottom, with a curved line connecting them. A blue dashed box encloses the top part of the diagram. The second diagram shows a white dot on top and a black dot on bottom, with a curved line connecting them. The third diagram shows a black dot on top and a white dot on bottom, with a curved line connecting them, but the blue dashed box encloses the bottom part of the diagram.

Proof.

A sequence of diagrammatic transformations proving the extended Frobenius law. The proof starts with a diagram of a white dot on top and a black dot on bottom. It then shows a series of steps where the diagram is modified by adding lines and dots, with blue dashed boxes highlighting the regions where the transformations occur. The final diagram is a simplified version of the original equation, showing a black dot on top and a white dot on bottom.

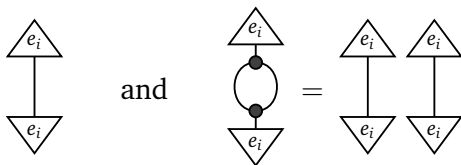
Speciality

If Ψ copies orthogonal basis $\{e_i\}$, can find (squared) norm of e_i :



Speciality

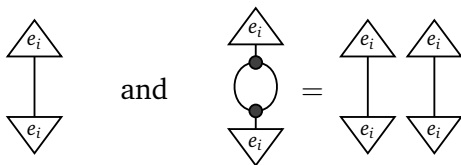
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So can characterize orthonormality via Frobenius structure.

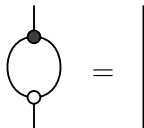
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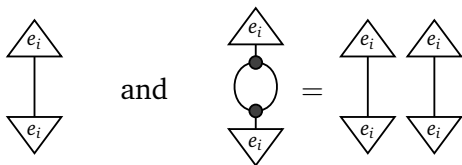
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A Frobenius structure is **special** if:



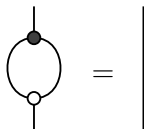
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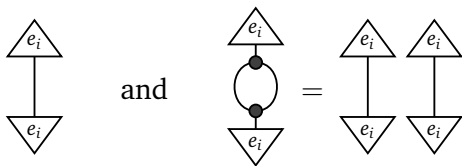
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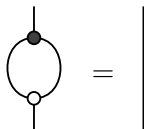
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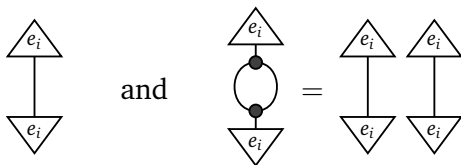


Examples:

- ▶ Group algebra in **FHilb** is only special for trivial group

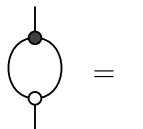
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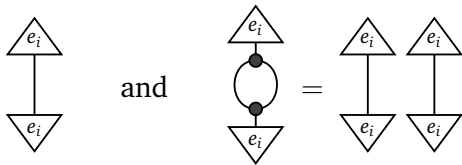


Examples:

- ▶ Group algebra in **FHilb** is only special for trivial group
- ▶ Orthogonal basis in **FHilb** is special just when basis is orthonormal

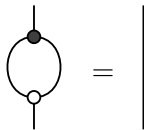
Speciality

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Examples:

- ▶ Group algebra in **FHilb** is only special for trivial group
- ▶ Orthogonal basis in **FHilb** is special just when basis is orthonormal
- ▶ Groupoid Frobenius structure in **Rel** is always special

Classical structures

In a braided monoidal dagger category, a **classical structure** is a special commutative dagger Frobenius structure.

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Definition of classical structure redundant:

- ▶ (Co)commutativity implies half of (co)unitality
- ▶ Speciality and Frobenius law imply (co)associativity
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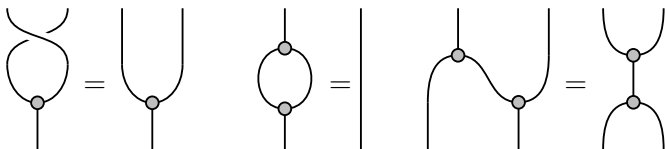
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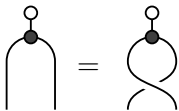
To check that $(A, \eta, \epsilon, \circlearrowleft, \circlearrowright)$ is classical structure, only need:



Symmetry

Pair of pants hardly ever commutative. However:

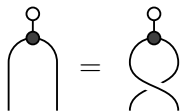
A Frobenius structure is **symmetric** when:



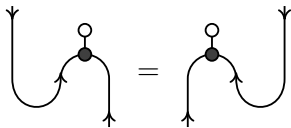
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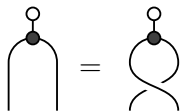


In a compact category, this is equivalent to the following:

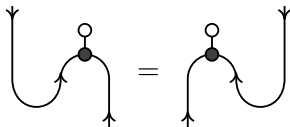


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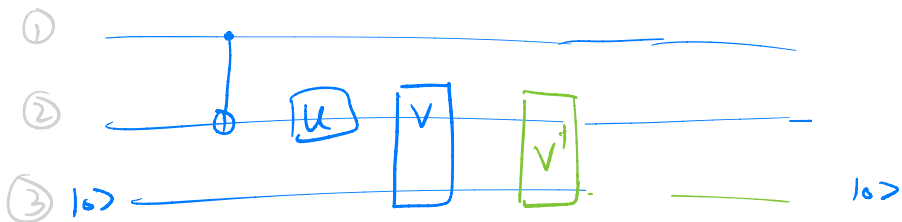
- ▶ Pair of pants: in **FHilb** this says $\text{Tr}(ab) = \text{Tr}(ba)$
- ▶ Group algebras: inverses in groups are two-sided inverses
- ▶ Groupoid Frobenius structure: inverses are two-sided

Self-duality

If $(A, \psi, \varphi, \bullet, \circ)$ Frobenius structure in monoidal category, then $A \dashv A$ is self-dual with:

A diagrammatic equation showing the self-duality of the multiplication and comultiplication. On the left, a cup-shaped arc with two inputs labeled A and one output labeled A . This is equal to a cap-shaped arc with two inputs labeled A and one output labeled A , with a solid black dot at the bottom vertex.

A diagrammatic equation showing the self-duality of the comultiplication and multiplication. On the left, a cap-shaped arc with two inputs labeled A and one output labeled A . This is equal to a cup-shaped arc with two inputs labeled A and one output labeled A , with an open white circle at the top vertex.



Self-duality

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$$\begin{array}{c} A \quad A \\ \cup \\ \text{---} \\ \cup \\ A \quad A \end{array} = \begin{array}{c} A \quad A \\ \cup \\ \circ \\ \bullet \\ A \quad A \end{array} \qquad \begin{array}{c} \text{---} \\ \cap \\ A \quad A \end{array} = \begin{array}{c} \circ \\ \bullet \\ \cap \\ A \quad A \end{array}$$

Proof.

$$\begin{array}{c} \text{---} \\ \cup \\ \text{---} \\ \cap \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \cup \\ \circ \\ \bullet \\ \text{---} \\ \cap \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \cup \\ \circ \\ \bullet \\ \text{---} \\ \cap \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

Nondegenerate forms

Monoid $(A, \triangleleft, \triangleright)$ forms Frobenius structure with comonoid (A, φ, ψ) iff allows **nondegenerate form**: map $\varphi: A \rightarrow I$ with



part of self-duality $A \dashv A$.

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Proof. One direction is the previous theorem.

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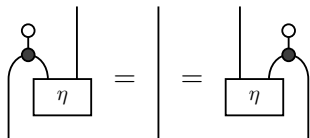
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Conversely, suppose $I \xrightarrow{\eta} A \otimes A$ satisfies:



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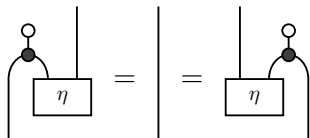
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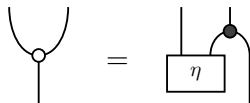
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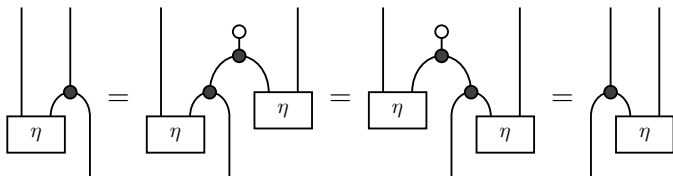
Define comultiplication as:



Nondegenerate forms

Proof (continued.)

Could have defined the comultiplication with η left or right:



Nondegenerate forms

Proof (continued.)

Could have defined the comultiplication with η left or right:

The diagram shows a sequence of four equivalent expressions for the product of two elements, each with a comultiplication element η attached to its left side. The first expression is a simple multiplication of two elements. The second expression shows the left element's comultiplication element η connected to the multiplication dot. The third expression shows the right element's comultiplication element η connected to the multiplication dot. The fourth expression shows the multiplication dot connected to the right element's comultiplication element η . A blue asterisk $(*)$ is placed to the right of the sequence.

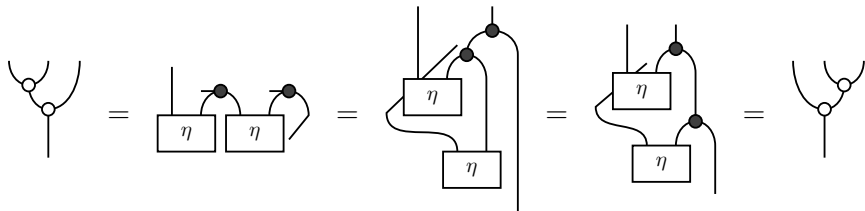
Counitality:

The diagram illustrates the counitality property. It starts with a comultiplication element η (a cup with a dot) connected to a multiplication element (a dot with a tail). This is shown to be equal to a diagram where the comultiplication element is connected to the multiplication element via a box labeled η . A blue box highlights this intermediate step, with a blue asterisk $(*)$ next to it. The final result is the original comultiplication element η .

Nondegenerate forms

Proof (continued.)

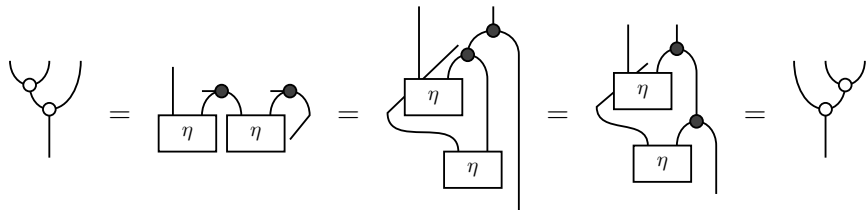
Coassociativity:



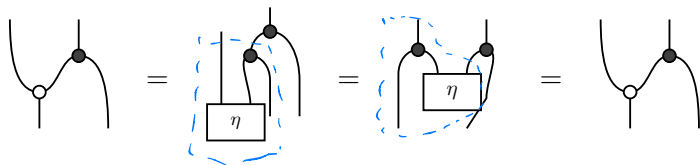
Nondegenerate forms

Proof (continued.)

Coassociativity:



Frobenius law:



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A **homomorphism of Frobenius structures** is morphism which is both monoid and comonoid homomorphism.

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Indeed:

Normal forms

Two ways to think about graphical calculus:

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merely shorthand to write down e.g. linear map;
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can be manipulated by replacing equal parts.

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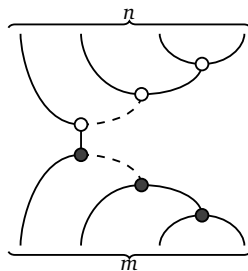
A **normal form** theorem connects the two:

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Unique way to copy (φ), discard (\emptyset), fuse (\cup), create (\circ) data!

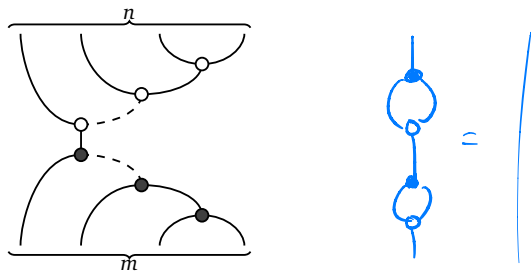
Spider theorem

Let $(A, \mu, \nu, \psi, \varphi)$ be a special Frobenius structure. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built out of finitely many pieces μ, ν, ψ, φ , and id , using \circ and \otimes , equals:



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Proof. Induction on the number of dots.

Spider theorem

Proof. (continued.)

Base case. Trivial, as the diagram must be one of \downarrow , \uparrow , \circlearrowleft , \circlearrowright .

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Induction step. Assume all diagrams with at most n dots can be brought in normal form, and consider a diagram with $n + 1$ dots.

Spider theorem

Proof. (continued.)

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Use naturality to write diagram in form with topmost dot.

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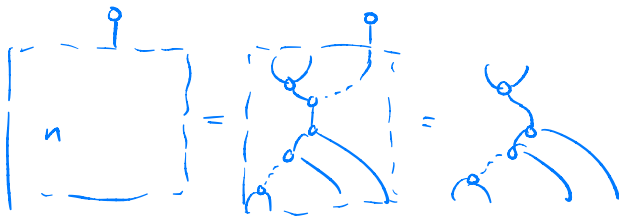
Proof. (continued.)

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- ▶ Topmost dot is φ : use counitality to eliminate it.



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- ▶ Topmost dot is ϕ : use counitality to eliminate it.
- ▶ Topmost dot is φ : use coassociativity to reach normal form.



Spider theorem

Proof. (continued.)

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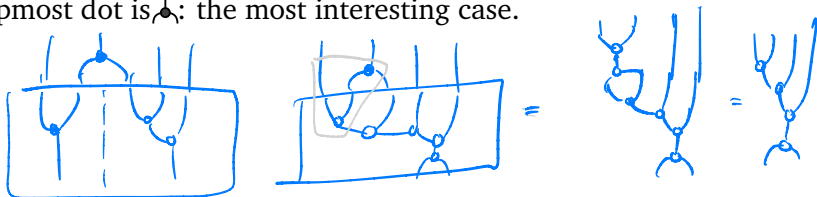
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Use naturality to write diagram in form with topmost dot.

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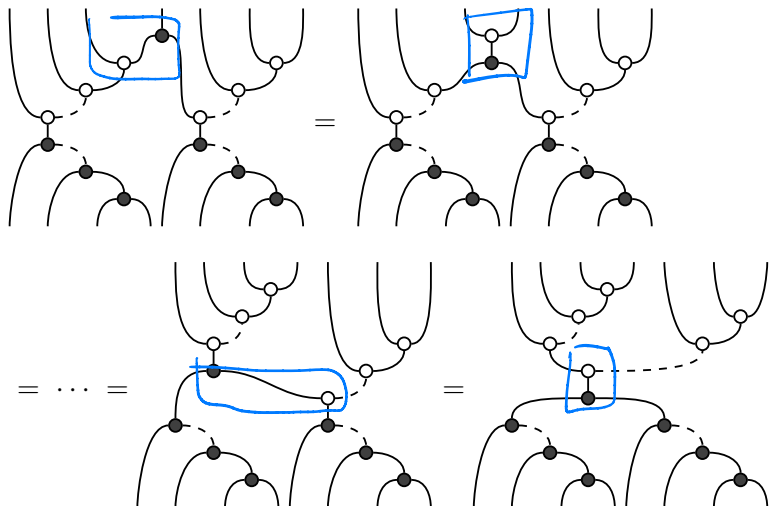
Is the diagram underneath the \circ connected?

If so, use coassociativity and speciality.

Spider theorem

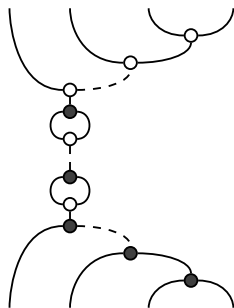
Proof. (continued.)

Suppose instead the rest of the diagram is disconnected:



More spider theorems

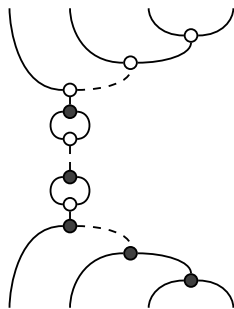
In a monoidal category, let $(A, \mu, \nu, \varphi, \rho)$ be a Frobenius structure. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built out of finitely many pieces μ, ν, φ, ρ , and id , using \circ and \otimes , equals $(*)$.



$(*)$

More spider theorems

In a monoidal category, let $(A, \mu, \nu, \varphi, \rho)$ be a Frobenius structure. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built out of finitely many pieces μ, ν, φ, ρ , and id , using \circ and \otimes , equals $(*)$.



$(*)$

In a symmetric monoidal category, let $(A, \mu, \nu, \varphi, \rho)$ be a commutative Frobenius structure. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built out of finitely many pieces μ, ν, φ, ρ , id , \bowtie , using \circ and \otimes , equals $(*)$.

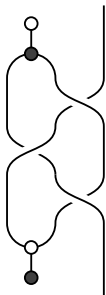
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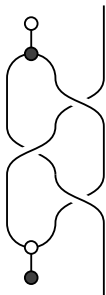
Proof. Regard the following diagram as a piece of string on which an overhand knot is tied:



No braided spider theorem

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Proof. Regard the following diagram as a piece of string on which an overhand knot is tied:



The Frobenius algebra axioms induce homotopy equivalences ('deformations') of the corresponding graph. Such moves cannot untie the knot.

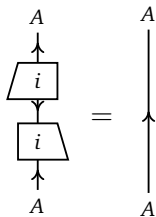
Involutive monoids

If (A, m, u) is monoid, so is (A^*, m_*, u_*) .

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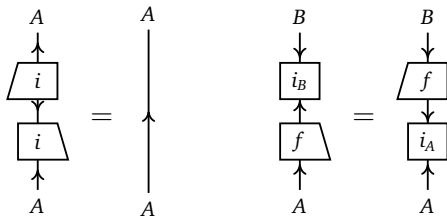
An **involution** for a monoid (A, \cdot, \circ) is a monoid homomorphism $A \xrightarrow{i} A^*$ satisfying $i_* \circ i = \text{id}_A$.



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An **involution** for a monoid (A, m, u) is a monoid homomorphism $A \xrightarrow{i} A^*$ satisfying $i_* \circ i = \text{id}_A$.



A **morphism of involutive monoids** is monoid homomorphism $A \xrightarrow{f} B$ satisfying $i_B \circ f = f_* \circ i_A$.

Example involutive monoids

- ▶ **Matrix algebra.** \mathbb{M}_n is an involutive monoid in **FHilb**.
Opposite monoid \mathbb{M}_n^* : multiplication ab in \mathbb{M}_n^* is ba in \mathbb{M}_n .
Canonical involution $\mathbb{M}_n \rightarrow \mathbb{M}_n^*$ given by $f \mapsto f^\dagger$.

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- ▶ **Pair of pants.** $A^* \otimes A$ involutive in a dagger pivotal category.
Identity map as involution, because of conventions:

$$\left(\begin{array}{c} \diagup \quad \diagdown \\ \cap \end{array} \right)_* = \left(\begin{array}{c} \text{U-shaped diagram with arrows} \end{array} \right)^\dagger = \begin{array}{c} \diagup \quad \diagdown \\ \cap \end{array}$$

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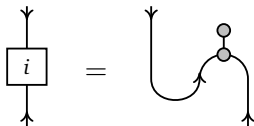
$$\left(\begin{array}{c} \diagup \quad \diagdown \\ \cap \end{array} \right)_* = \left(\begin{array}{c} \text{U-shaped loop with arrows} \\ \text{with a dot on the top line} \end{array} \right)^\dagger = \begin{array}{c} \diagup \quad \diagdown \\ \cap \end{array}$$

- ▶ **Groupoids.** \mathbf{G} in **Rel** is involutive.
 Opposite monoid: induced by opposite groupoid \mathbf{G}^{op}

Canonical involution $G \rightarrow G^*$ given by $g \sim g^{-1}$.

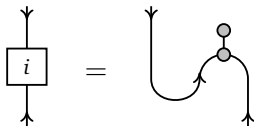
Frobenius law from way of the dagger

Monoid (A, \circ, \circ) is dagger Frobenius if and only if i is involution:



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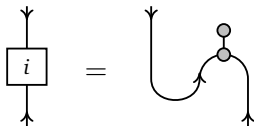
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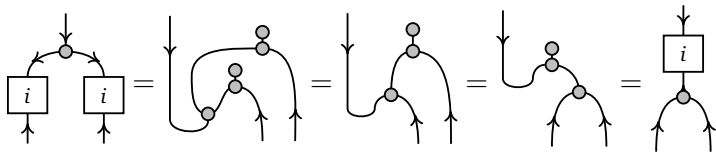
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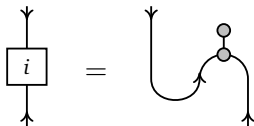
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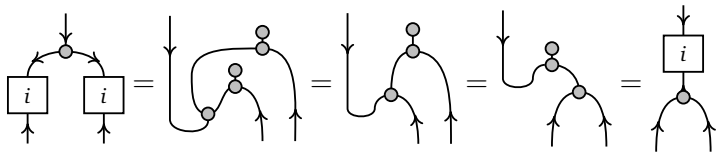
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Monoid (A, \cdot, \circ) is dagger Frobenius if and only if i is involution:


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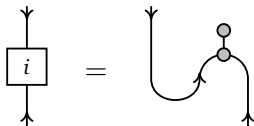
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► i preserves units: easy.

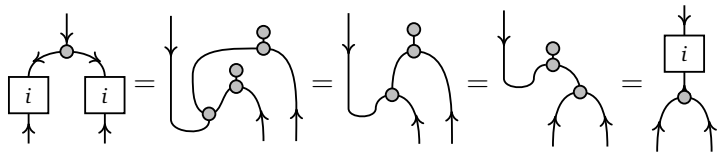
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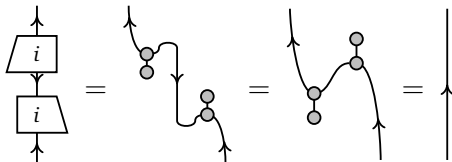
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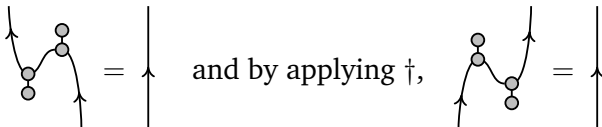
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Proof. (continued.) Conversely, suppose $i_* \circ i = \text{id}$. Then:

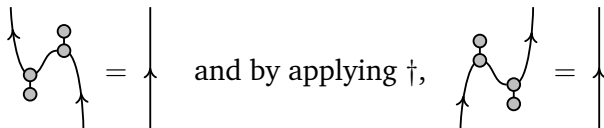


The diagram shows two equations. The first equation shows a diagram with three nodes and two lines. The left line starts at the top, goes down to the first node, then up to the second node, then down to the third node, and finally up to the top. The right line starts at the bottom, goes up to the third node, then down to the second node, then up to the first node, and finally down to the bottom. This diagram is equal to a single vertical line with an upward arrow. The second equation is identical to the first, but the nodes are arranged in a different order, and it is preceded by the text "and by applying †,".

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Frobenius law from way of the dagger

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The diagram shows two equations. The first equation shows a curved line with two nodes (circles) on the left side, connected to a straight vertical line on the right side, with an equals sign between them. The second equation shows a similar curved line with two nodes on the right side, connected to a straight vertical line on the left side, also with an equals sign between them. The text "and by applying †," is placed between the two equations.

So we have a Frobenius structure, defined by a nondegenerate form.
Is it a dagger Frobenius structure?

Frobenius law from way of the dagger

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The diagram shows two equations. The first equation shows a diagram with a vertical line on the left, a loop with two nodes, and a vertical line on the right, equated to a single vertical line. The second equation shows a similar diagram with the loop mirrored, also equated to a single vertical line. The text "and by applying †," is placed between the two equations.

So we have a Frobenius structure, defined by a nondegenerate form. Is it a dagger Frobenius structure?

The condition that i preserves multiplication gives:

The diagram shows a sequence of three equations. The first equation shows a complex diagram with a loop and a vertical line, equated to a simpler diagram. The second equation shows a similar diagram, equated to a diagram with a loop and a vertical line. The third equation shows a diagram with a loop and a vertical line, equated to a diagram with a loop and a vertical line.

So the form definition gives rise to the correct comultiplication.

Classification in **FHilb**

Theorem: special dagger Frobenius structures in **FHilb** are of the form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$.

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- ▶ Cayley: dagger Frobenius structure on H embeds into $H^* \otimes H$
- ▶ $H^* \otimes H$ isomorphic to $\mathbb{M}_{\dim(H)}$
- ▶ so H involutive subalgebra of $\mathbb{M}_{\dim(H)}$: C^* -algebra
- ▶ Artin-Wedderburn: must be of form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$

Classification in **FHilb**

$$(a_1, \dots, a_k) \in A \quad a_i: \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_i}$$
$$(L_1, \dots, L_k)$$

$$(a_1, \dots, a_k) \cdot (L_1, \dots, L_k) = (a_1 L_1, \dots, a_k L_k)$$

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Corollary: classical structure in **FHilb** copy orthonormal bases

Proof: must be of form $\mathbb{C} \oplus \dots \oplus \mathbb{C} = \mathbb{C}^n$

$$\psi(x_1, \dots, x_k) = \sum x_i e_i$$

$$\psi((x_1, \dots, x_k) \otimes (y_1, \dots, y_k)) = x_1 y_1 \otimes \dots \otimes x_k y_k$$

M_n commutative $\Leftrightarrow n=1$

$$\Rightarrow: a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$aL \neq La$$

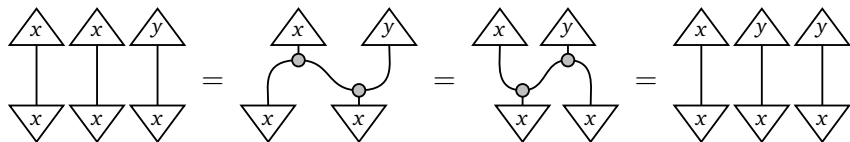
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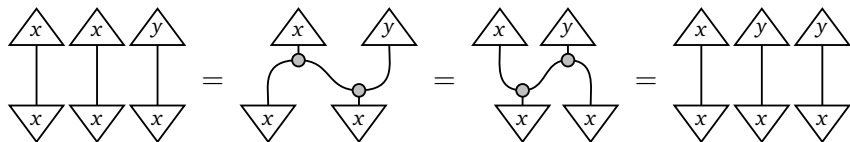
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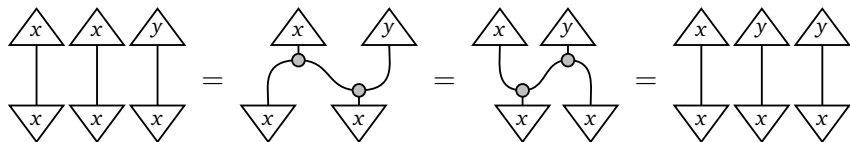


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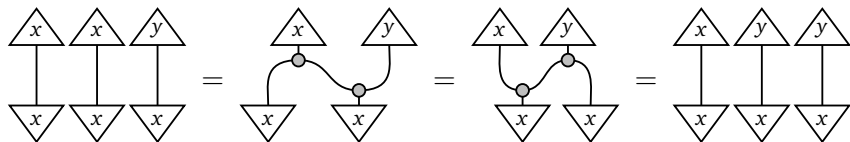
If $\langle x|y \rangle = 0$, then this is satisfied.

If $\langle x|y \rangle \neq 0$, this implies $\langle x|x \rangle = \langle x|y \rangle$. Similarly $\langle y|x \rangle = \langle y|y \rangle$.

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If $\langle x|y \rangle \neq 0$, this implies $\langle x|x \rangle = \langle x|y \rangle$. Similarly $\langle y|x \rangle = \langle y|y \rangle$.

Hence $\langle x - y|x - y \rangle = \langle x|x \rangle - \langle x|y \rangle - \langle y|x \rangle + \langle y|y \rangle = 0$, so $x = y$.

Orthogonal bases and morphisms

In **FHilb**, morphism between two commutative dagger Frobenius structures acts as function on copyable states if and only if it is comonoid homomorphism.

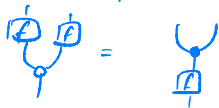
A function

$$f = \{d_1, \dots, d_n\} \rightarrow \{e_1, \dots, e_n\}$$

induces a linear function

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ |d_i\rangle & \longmapsto & |f(d_i)\rangle \end{array}$$

that is in fact a comonoid homomorphism



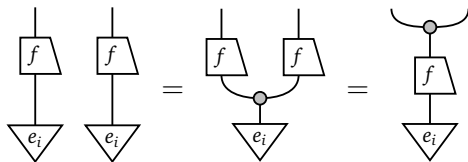
where $\psi : |d_i\rangle \mapsto |d_i\rangle \otimes |d_i\rangle$

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Orthogonal bases and morphisms

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Proof. Suffices to see about basis of copyable states $\{e_i\}$.



Hence $f(e_i)$ copyable.

Classification in **Rel**

Theorem: Special dagger Frobenius structures in **Rel** correspond exactly to groupoids.

Classification in **Rel**

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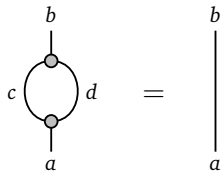
Proof. Write $A \times A \xrightarrow{M} A$ for multiplication, $U \subseteq A$ for unit.

Classification in **Rel**

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Proof. Write $A \times A \xrightarrow{M} A$ for multiplication, $U \subseteq A$ for unit.

M is single-valued: by speciality $a(M \circ M^\dagger)b$ iff $a = b$:



So: if $(c, d)Ma$ and $(c, d)Mb$, must have $a = b$.

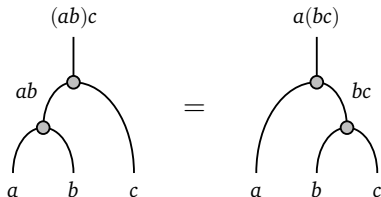
May simply write ab for unique c with $(a, b)Mc$.

Remember: ab not always defined!

Classification in Rel

Proof. (continued)

Associativity:



So ab and $(ab)c$ defined exactly when bc and $a(bc)$ are defined, and then $(ab)c = a(bc)$.

Classification in Rel

Proof. (continued)

Unitality: for units $x, y \in U$

$$\begin{array}{c} b \\ | \\ \bullet \\ \swarrow \quad \searrow \\ x \quad a \end{array} = \begin{array}{c} b \\ | \\ a \end{array} = \begin{array}{c} b \\ | \\ \bullet \\ \swarrow \quad \searrow \\ a \quad y \end{array}$$

So: a, b allow $x \in U$ with $xa = b$ iff $a = b$.

And: a, b allow $y \in U$ with $ay = b$ iff $a = b$.

If $z \in U$ then $xz = x$ for some $x \in U$. But then $x = z$!

Units idempotent; multiplication of different ones undefined.

If $xa = a = x'a$, then $a = xa = x(x'a) = (xx')a$, so $x = x'$.

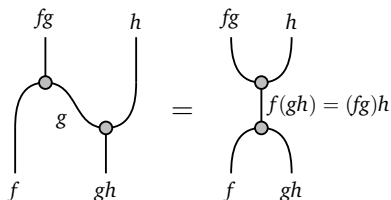
So every element has unique left/right identity.

Classification in Rel

Proof. (continued)

Category: U set of objects, A set of morphisms.

If fg defined and gh defined, want $(fg)h = f(gh)$ defined too:

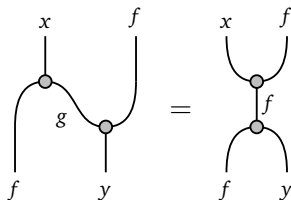


If fg and gh defined then LHS defined, so RHS defined too.

Classification in Rel

Proof. (continued)

Inverses: for $f \in A$ with left unit x and right unit y :

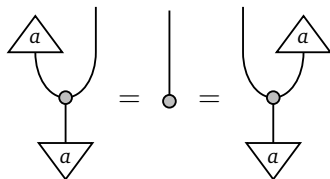


$$\left. \begin{aligned} gf = x &= id_{\text{dom}(f)} \\ fg = y &= id_{\text{cod}(f)} \end{aligned} \right\} \Rightarrow g = f^{-1}$$

□

Phases

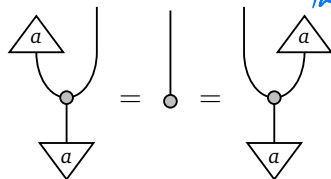
Let (A, \otimes, \oplus) be Frobenius structure in a monoidal dagger category.
State $I \xrightarrow{a} A$ is called **phase** when:



Phases

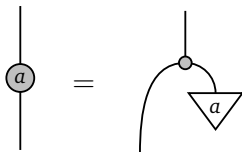
Let (A, α, β) be Frobenius structure in a monoidal dagger category.

State $I \xrightarrow{a} A$ is called **phase** when:



*if ψ copies basis $|e_i\rangle$
then*
 $b = \sum e_i$ $a = \sum a_i e_i$
 $\begin{matrix} \triangle \\ | \\ \triangle \end{matrix} = \sum_i a_i \begin{matrix} \triangle \\ | \\ \triangle \end{matrix} \begin{matrix} \triangle \\ | \\ \triangle \end{matrix}$
 $= \sum_i a_i \langle a | e_i \rangle e_i$

Its (right) **phase shift** is the following morphism $A \rightarrow A: = \sum_i a_i a_i^* \langle e_j | e_i \rangle \mathcal{D}_e$



$= \sum_i a_i a_i^* e_i$
 $= \sum_i |a_i|^2 e_i$

Example phases

- ▶ For classical structure in **FHilb** copying basis $\{e_i\}$, vector $a = a_1e_1 + \cdots a_ne_n$ is phase iff each a_i on unit circle: $|a_i|^2 = 1$.

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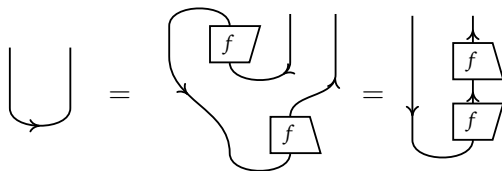
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Proof. The name of a morphism $A \xrightarrow{f} A$ is a phase when:



But this means $f \circ f^\dagger = \text{id}_A$; similarly $f^\dagger \circ f = \text{id}_A$.

Example phases

- ▶ Phases of Frobenius structure \mathbb{M}_n in **FHilb** form set $U(n)$ of n -by- n unitary matrices. Hence phases of $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ range over $U(k_1) \times \cdots \times U(k_n)$.

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- ▶ The phases of a Frobenius structure in **Rel** induced by a group G are elements of that group G itself.

Proof. For a subset $a \subseteq G$, equation defining phases reads

$$\{g^{-1}h \mid g, h \in a\} = \{1_G\} = \{hg^{-1} \mid g, h \in a\}.$$

So if $g \in G$, then $a = \{g\}$ is a phase. But if a contains distinct elements $g \neq h$ of G , cannot be phase. Similarly, $a = \emptyset$ not phase. Hence a phase precisely when singleton $\{g\}$.

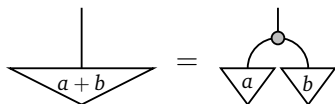
Phase group

In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit \circ and:

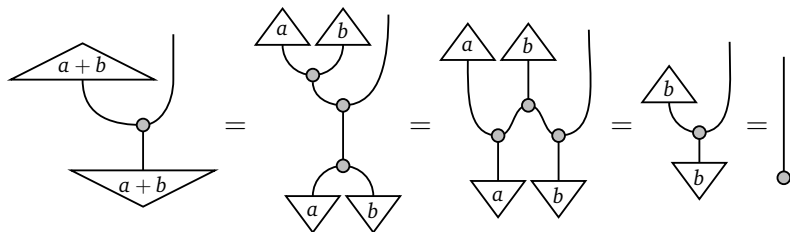
The diagram shows an equality between two expressions. On the left, a vertical line descends from the top and meets the top vertex of a downward-pointing triangle labeled $a + b$. On the right, a vertical line descends from the top and meets a small grey circle. From this circle, two curved lines arc downwards to the top vertices of two separate downward-pointing triangles labeled a and b . An equals sign is placed between the two expressions.

Phase group

In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit \circ and:


$$\text{triangle}(a+b) = \text{dot} \text{ over } (\text{triangle}(a) \text{ and } \text{triangle}(b))$$

Proof. This is again a well-defined phase:


$$\text{triangle}(a+b) \text{ over } \text{triangle}(a+b) = \text{triangle}(a) \text{ and } \text{triangle}(b) \text{ over } \text{triangle}(a) \text{ and } \text{triangle}(b) = \text{triangle}(a) \text{ and } \text{triangle}(b) \text{ over } \text{triangle}(a) \text{ and } \text{triangle}(b) = \text{triangle}(b) \text{ over } \text{triangle}(b) = \text{dot}$$

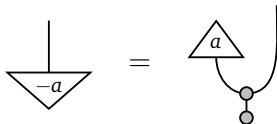
The flipped equation follows similarly.

Associativity is clear, hence phases form a monoid.

Phase group

Proof. (continued)

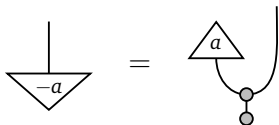
Left-inverse of phase a is:



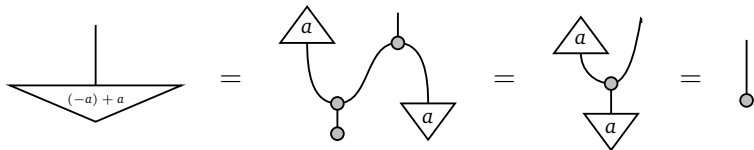
Phase group

Proof. (continued)

Left-inverse of phase a is:



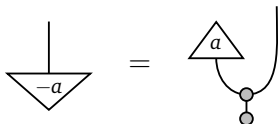
Left-inverse of a is $-a$:



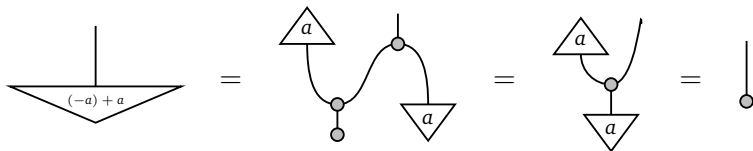
Phase group

Proof. (continued)

Left-inverse of phase a is:



Left-inverse of a is $-a$:



Similarly there is right-inverse. But in monoids, left and right inverses are equal: $l = l(xr) = (lx)r = r$.



Example phase groups

- ▶ In **FHilb**, the phase group for the pair of pants Frobenius structure is the unitary group.

Example phase groups

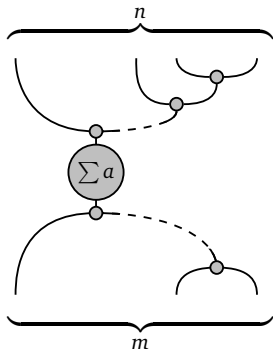
- ▶ In **FHilb**, the phase group for the pair of pants Frobenius structure is the unitary group.
- ▶ Phase addition in the Frobenius structure $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ in **FHilb** is entrywise multiplication in $U(k_1) \times \cdots \times U(k_n)$. In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.

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- ▶ In **Rel**, the phase group induced by a group G is the group itself.

Phased spider theorem

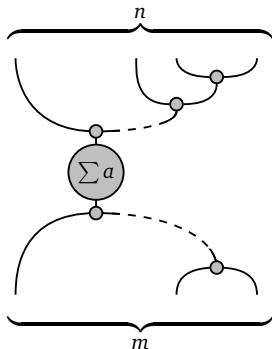
Let $(A, \multimap, \circ, \oplus)$ be classical structure in braided monoidal dagger category. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built of finitely many $\multimap, \circ, \oplus, \text{id}, \sigma$ and phases using \circ, \otimes , and \dagger , equals



where a ranges over all the phases used in the diagram.

Phased spider theorem

Let $(A, \circlearrowleft, \circlearrowright)$ be classical structure in braided monoidal dagger category. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built of finitely many $\circlearrowleft, \circlearrowright, \text{id}, \sigma$ and phases using \circ, \otimes , and \dagger , equals



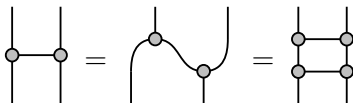
where a ranges over all the phases used in the diagram.

Proof. Use braidings to have all phases dangle at the bottom. Apply Spider Theorem. Use phase addition to reduce to single phase $\sum a$ on bottom right. Apply Spider Theorem again. □

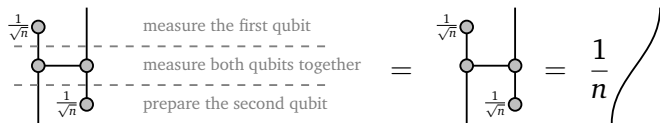
State transfer

State transfer protocol: transfer state of Hilbert space H from one system to another, with success probability $1/\dim(H)^2$.

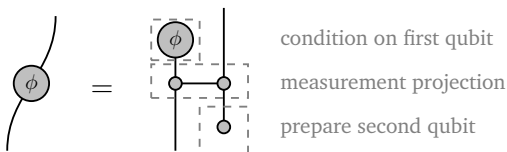
May be lax in drawing, e.g. projection $H \otimes H \rightarrow H \otimes H$:



The procedure looks like this:



Extra challenge: apply phase gate while transferring state



Problem:

ρ is not really a measurement because it uses
pure states $|\psi\rangle \in \mathbb{C}^2$ whereas it needs
mixed states $\sum_i |\psi_i\rangle\langle\psi_i| : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ "density matrices"
 because measuring one half of a bipartite pure state
 is generally a mixed state.

Answer:

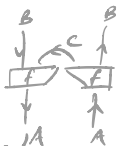
don't work in $\mathbb{F}H_i/B$, but $\text{CPM}(\mathbb{F}H_i/B)$.

if C is dagger compact, so is $\text{CPM}(C)$.

So can use same pictures, but also distinguish
 "quantum wires" and "classical wires".

a state $I \rightarrow I$ in $\mathbb{F}H_i/B$ is a pure state (like $|\psi\rangle$)

_____ $\text{CPM}(\mathbb{F}H_i/B)$ _____ mixed state (like $\sum_{i=1}^n |\psi_i\rangle\langle\psi_i|$)



Summary

- ▶ Frobenius structures: interacting co/monoid, self-duality
- ▶ Normal forms: spider theorems
- ▶ Frobenius law: justified by coherence
- ▶ Classification: matrix algebras, bases, groupoids
- ▶ Phases: unitary operators, state transfer

Next week: interaction between two Frobenius structures