Introduction to Quantum Programming and Semantics

Week 7: Frobenius structures

Chris Heunen

THE UNIVERSITY *of* EDINBURGH
informatics

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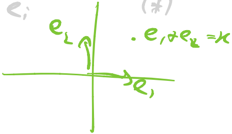
Overview

- Frobenius structure: interacting co/monoid, self-duality
- Normal forms: coherence theorem
- Frobenius law: coherence between dagger and closure
- Classification: in **FHilb** and **Rel**
- Phases: unitary operators

Idea

Planchorel: $x = \sum_i \langle e_i | x \rangle e_i$ (*)



Orthonormal basis $\{e_i\}$ for H in **FHilb** gives comonoid (copy): $e_i \mapsto e_i \otimes e_i$.
Its adjoint (merge) is comparison: $e_i \otimes e_i \mapsto e_i$ and $e_i \otimes e_j \mapsto 0$ if $i \neq j$.

$$\langle f(x) | y \rangle = \langle x | f^\dagger(y) \rangle$$

$$\langle \text{copy}(e_i) | e_j \otimes e_k \rangle = \langle e_i | \text{merge}(e_j \otimes e_k) \rangle$$

$$\| \quad \langle e_i \otimes e_i | e_j \otimes e_k \rangle$$

$$\| \quad \langle e_i | e_j \rangle \langle e_i | e_k \rangle$$

$$\| \quad \begin{cases} 1 & \text{iff } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

so if $j = k$: pick $i = j$ and:

$1 = \langle e_i | \text{merge}(e_i \otimes e_i) \rangle$

i.e. $e_i = \text{merge}(e_i \otimes e_i)$ (*)

or if $j \neq k$

$0 = \langle e_i | \text{merge}(e_j \otimes e_k) \rangle \ \forall i$

so $\text{merge}(e_j \otimes e_k) = 0$

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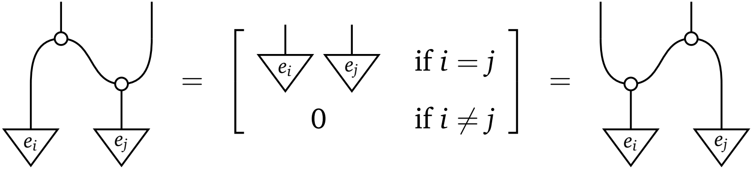

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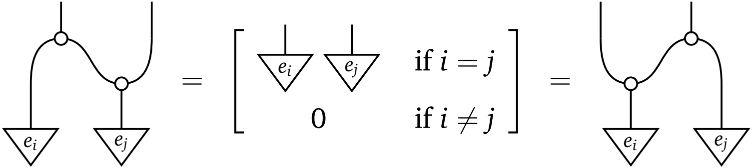
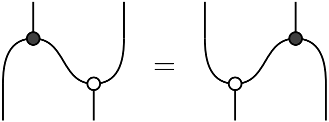


This monoid/comonoid interaction is called the Frobenius law.

Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid $(A,$ , $)$ and monoid $(A,$ , $)$ satisfying the Frobenius law:



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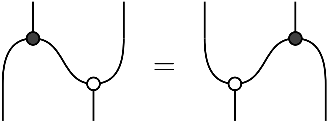

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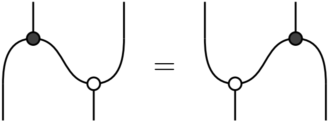

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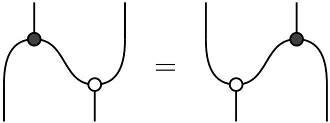

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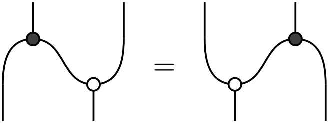

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  Adjoint: : $\sum_{h \in G} gh^{-1} \otimes h$, and : $1_G \mapsto g$ and $1_G \neq g \mapsto 0$.

$G = (\mathbb{Z}_3, +, 0)$

$A = \mathbb{C}^3$

$: \mathbb{C}^3 \longrightarrow \mathbb{C}$

$0 \neq |0\rangle \longmapsto 1$

$|1\rangle \longmapsto 0$

$|2\rangle \longmapsto 0$

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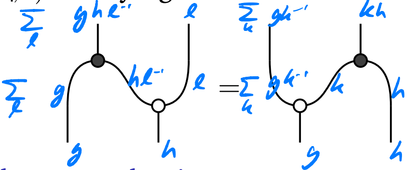

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  Frobenius law: $\text{LHS}(g \otimes h) = \sum_{k \in G} gk^{-1} \otimes kh = \text{RHS}(g \otimes h)$.

$k = \ell h^{-1}$
$\ell = kh$

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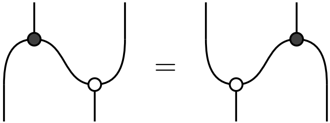

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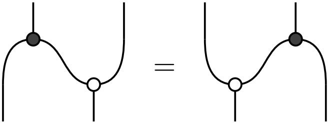

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  Frobenius law: $(g, h) \sim (a, b \circ h)$ for $g = a \circ b$, $\mathrm{t}(h) = \mathrm{s}(b)$.

Pair of pants

In a dagger monoidal category, if $A \dashv A^*$, the pair of pants monoid $A^* \otimes A$ carries a dagger Frobenius structure.

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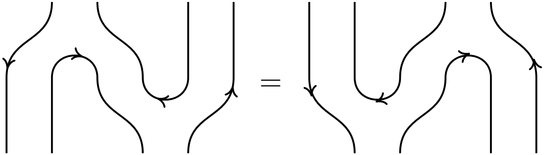
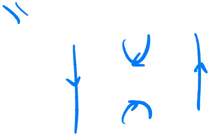
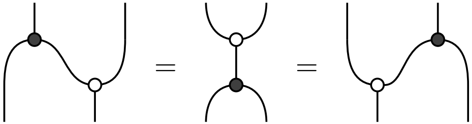
Proof.





Extended Frobenius law

Any Frobenius structure satisfies:



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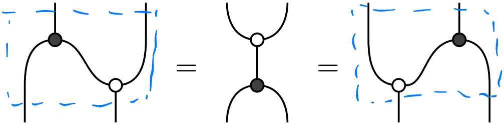

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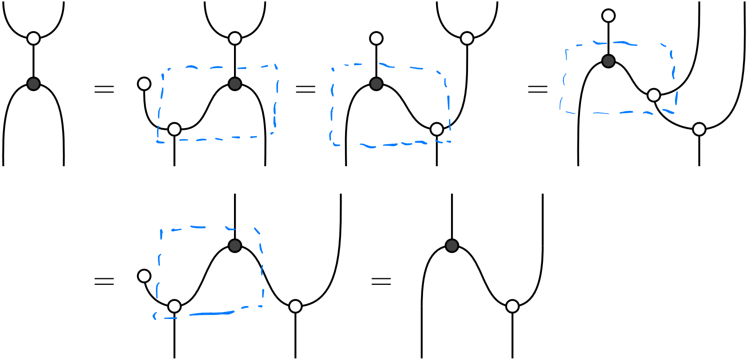
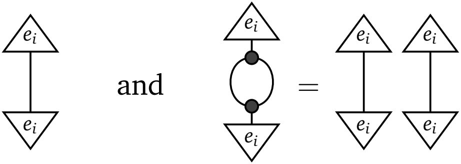


Speciality

If  copies orthogonal basis $\{e_i\}$, can find (squared) norm of e_i:



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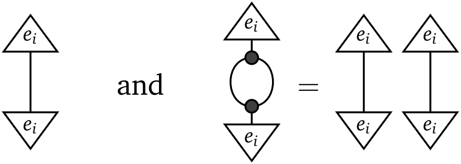

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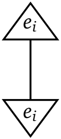

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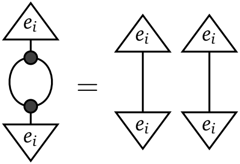

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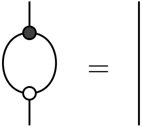

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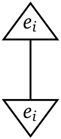

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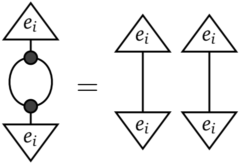

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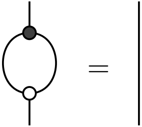

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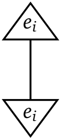

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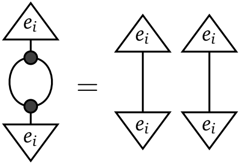

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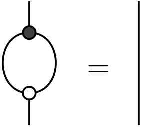

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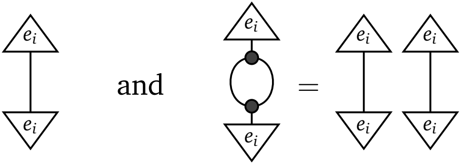

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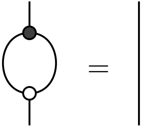

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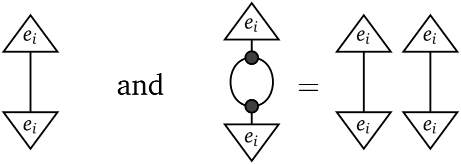

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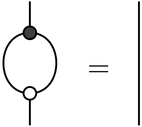

Examples:

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- Groupoid Frobenius structure in **Rel** is always special

Classical structures

In a braided monoidal dagger category, a classical structure is a special commutative dagger Frobenius structure.

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- (Co)commutativity implies half of (co)unitality
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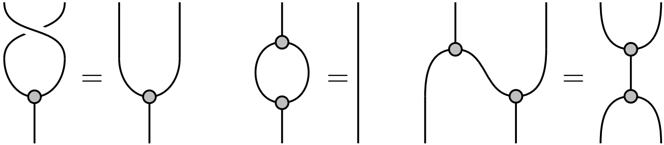
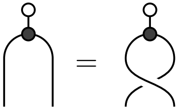
To check that (A, , ) is classical structure, only need:



Symmetry

Pair of pants hardly ever commutative. However:

A Frobenius structure is symmetric when:



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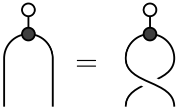

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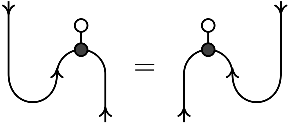

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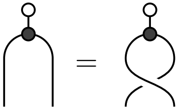

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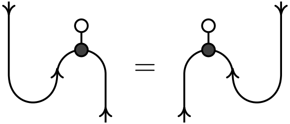
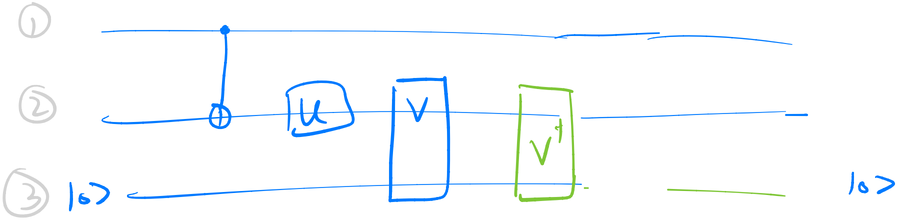


Examples:

- Pair of pants: in **FHilb** this says $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$
- Group algebras: inverses in groups are two-sided inverses
- Groupoid Frobenius structure: inverses are two-sided

Self-duality

If $(A,$  $)$ Frobenius structure in monoidal category, then $A \dashv A$ is self-dual with:


1
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V†
|0⟩

1
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1
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3
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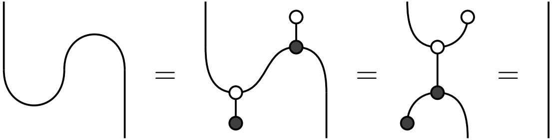
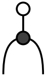


Nondegenerate forms

Monoid $(A, ,)$ forms Frobenius structure with comonoid $(A, ,)$ iff allows nondegenerate form: map $: A \to I$ with



part of self-duality $A \dashv A$.

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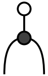

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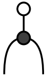

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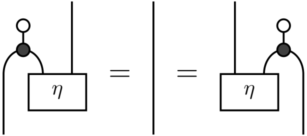
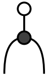
Conversely, suppose $I \xrightarrow{\eta} A \otimes A$ satisfies:



Nondegenerate forms

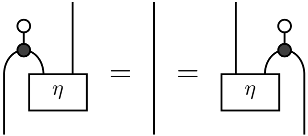
Monoid $(A, \curlywedge, \bullet)$ forms Frobenius structure with comonoid $(A, \curlyvee, \circ)$ iff allows nondegenerate form: map $\circ \colon A \to I$ with



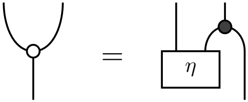
part of self-duality $A \dashv A$.

Proof. One direction is the previous theorem.

Conversely, suppose $I \xrightarrow{\eta} A \otimes A$ satisfies:



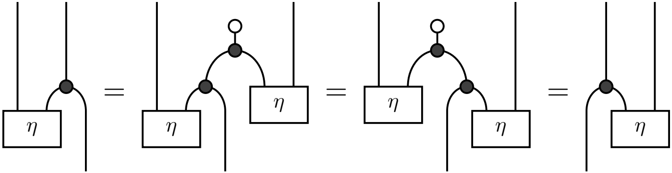
Define comultiplication as:



Nondegenerate forms

Proof (continued.)

Could have defined the comultiplication with η left or right:



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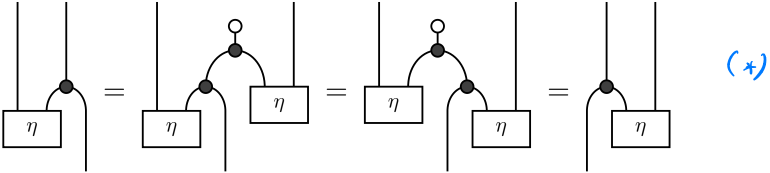
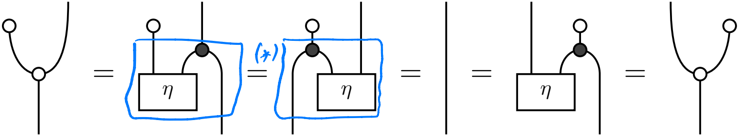


Counitality:



Nondegenerate forms

Proof (continued.)

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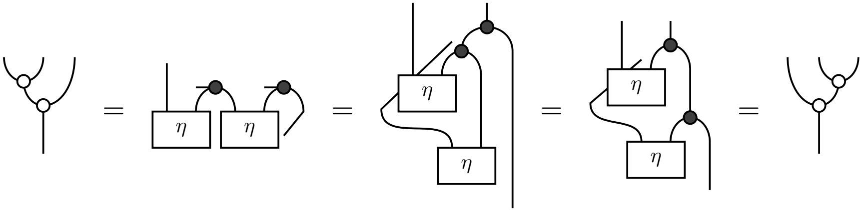

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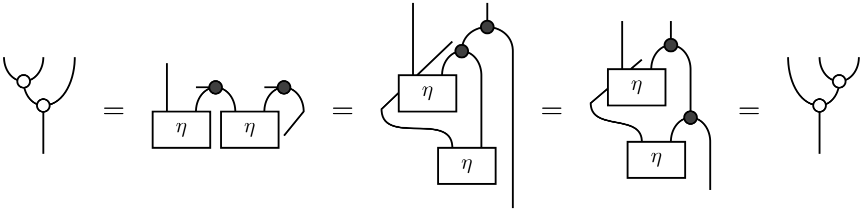
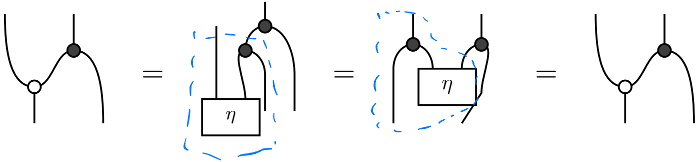


Frobenius law:



Homomorphisms

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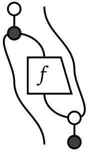
Proof. Given homomorphism $A \xrightarrow{f} B$, construct inverse as:

$f^{-1} =$ 

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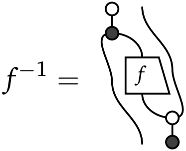
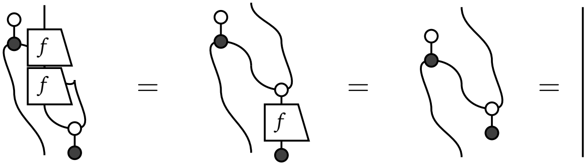


Indeed:



Normal forms

Two ways to think about graphical calculus:

- diagram represents morphism:
  merely shorthand to write down e.g. linear map;
- diagram is entity in its own right:
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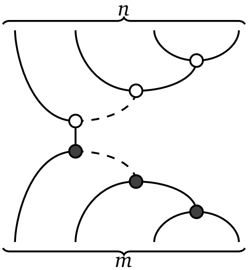
Unique way to copy (), discard (), fuse (), create () data!

Spider theorem

Let $(A,$, , , $)$ be a special Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , and id, using $\circ$ and $\otimes$, equals:



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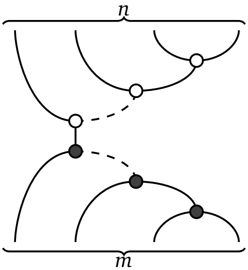

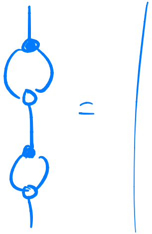

Proof. Induction on the number of dots.

Spider theorem

Proof. (continued.)

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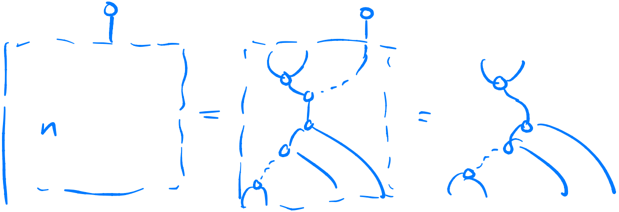
- Topmost dot is : use counitality to eliminate it.



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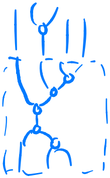

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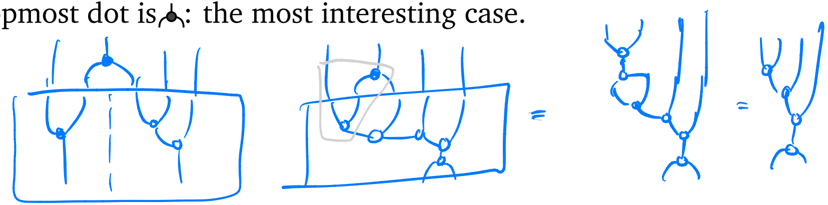

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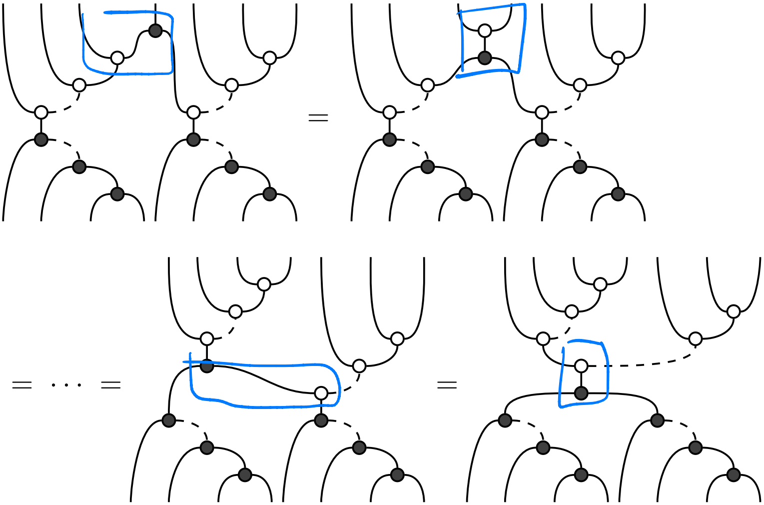
  Is the diagram underneath the connected?
  If so, use coassociativity and speciality.

Spider theorem

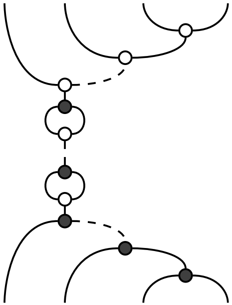
Proof. (continued.)

Suppose instead the rest of the diagram is disconnected:



More spider theorems

In a monoidal category, let $(A, , , ,)$ be a Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , and id, using $\circ$ and $\otimes$, equals $(*)$.



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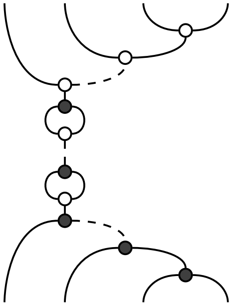

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In a symmetric monoidal category, let $(A, , , ,)$ be a commutative Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , id, , using $\circ$ and $\otimes$, equals $(*)$.

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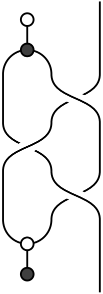

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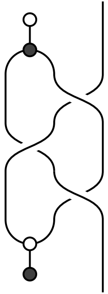

The Frobenius algebra axioms induce homotopy equivalences ('deformations') of the corresponding graph. Such moves cannot untie the knot.

Involutive monoids

If (A, m, u) is monoid, so is (A^*, m_*, u_*).

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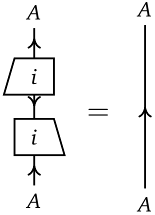
An involution for a monoid $(A,$ ⋏, ⫰$)$ is a monoid homomorphism $A \xrightarrow{i} A^*$ satisfying $i_* \circ i = \mathrm{id}_A$.



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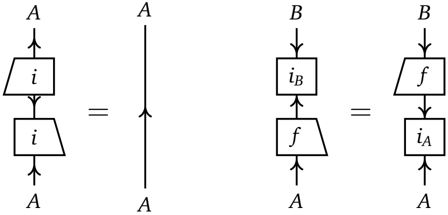

A morphism of involutive monoids is monoid homomorphism $A \xrightarrow{f} B$ satisfying $i_B \circ f = f_* \circ i_A$.

Example involutive monoids

- Matrix algebra. $\mathbb{M}_n$ is an involutive monoid in **FHilb**.
  Opposite monoid $\mathbb{M}_n^*$: multiplication ab in $\mathbb{M}_n^*$ is ba in $\mathbb{M}_n$.
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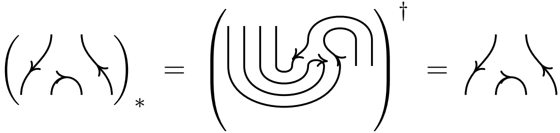

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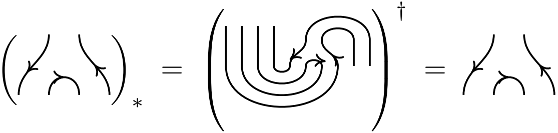
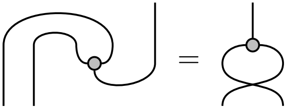


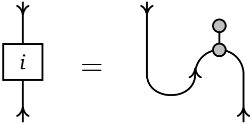
- Groupoids. **G** in **Rel** is involutive.
  Opposite monoid: induced by opposite groupoid $\mathbf{G}^{\mathrm{op}}$



Canonical involution $G \to G^*$ given by $g \sim g^{-1}$.

Frobenius law from way of the dagger

Monoid (A, , ) is dagger Frobenius if and only if i is involution:



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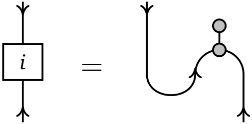

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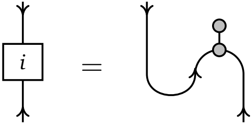

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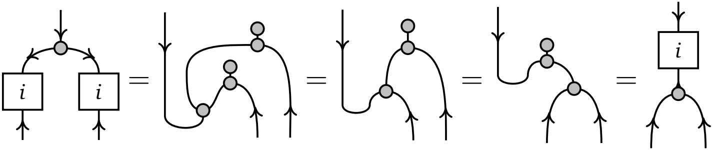

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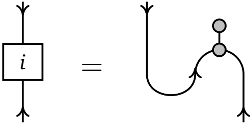

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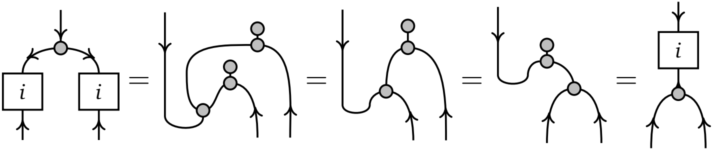

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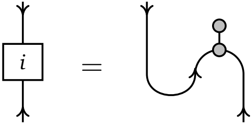

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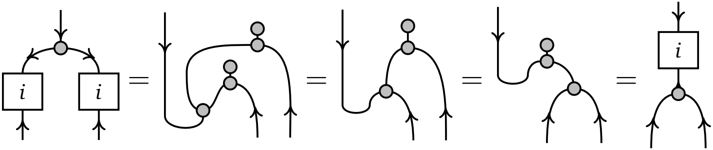

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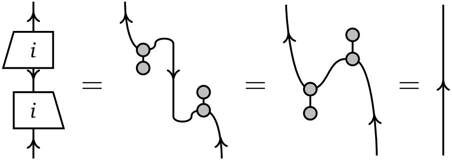

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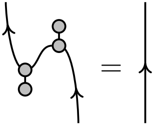
Proof. (continued.) Conversely, suppose $i_* \circ i = \mathrm{id}$. Then:



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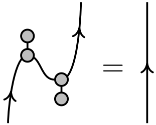

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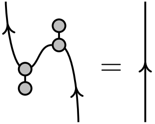

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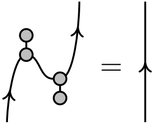

So we have a Frobenius structure, defined by a nondegenerate form.

Is it a dagger Frobenius structure?

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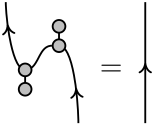

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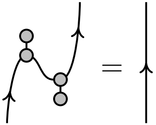

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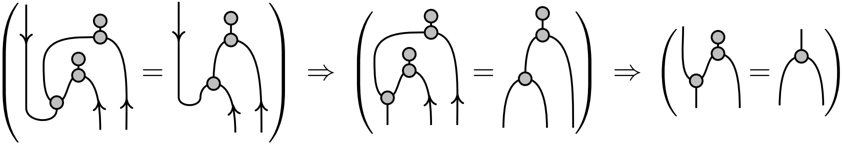

So the form definition gives rise to the correct comultiplication.

Classification in **FHilb**

Theorem: special dagger Frobenius structures in **FHilb** are of the form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$.

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Proof:

- Cayley: dagger Frobenius structure on H embeds into $H^* \otimes H$
- $H^* \otimes H$ isomorphic to $\mathbb{M}_{\dim(H)}$
- so H involutive subalgebra of $\mathbb{M}_{\dim(H)}$: C*-algebra
- Artin-Wedderburn: must be of form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$

Classification in **FHilb**

$(a_1,\dots,a_k)\in A$ $\quad a_i:\mathbb{C}^{n_i}\to\mathbb{C}^{n_i}$

$(b_1,\dots,b_k)$

$(a_1,\dots,a_k)\cdot(b_1,\dots,b_k) = (a_1b_1,\dots,a_kb_k)$

Theorem: special dagger Frobenius structures in **FHilb** are of the form $\mathbb{M}_{n_1}\oplus\cdots\mathbb{M}_{n_k}$. $= A$

Proof:

- Cayley: dagger Frobenius structure on H embeds into $H^*\otimes H$
- $H^*\otimes H$ isomorphic to $\mathbb{M}_{\dim(H)}$
- so H involutive subalgebra of $\mathbb{M}_{\dim(H)}$: C*-algebra
- Artin-Wedderburn: must be of form $\mathbb{M}_{n_1}\oplus\cdots\mathbb{M}_{n_k}$

Corollary: classical structure in **FHilb** copy orthonormal bases

Proof: must be of form $\mathbb{C}\oplus\cdots\oplus\mathbb{C}$. $=\mathbb{C}^n$

$\mathbb{M}_n$ commutative $\iff n=1$

$\varphi(x_1,\dots,x_k) = \sum x_i e_i$

$\Rightarrow$: $a=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\quad ab\neq ba$

$b=\begin{pmatrix}1&0\\0&0\end{pmatrix}$

$d((x_1,\dots,x_k)\otimes(y_1,\dots,y_k)) = x_1y_1\oplus\cdots\oplus x_ky_k$

Orthogonal bases

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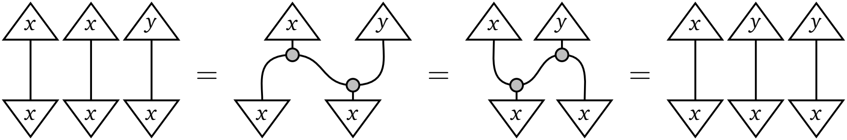

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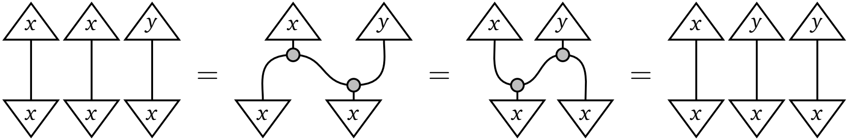

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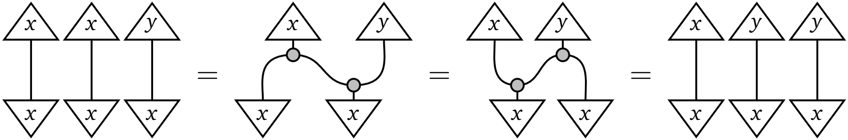

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If $\langle x|y\rangle \neq 0$, this implies $\langle x|x\rangle = \langle x|y\rangle$. Similarly $\langle y|x\rangle = \langle y|y\rangle$.

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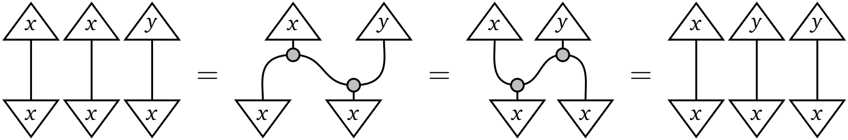

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Hence $\langle x-y|x-y\rangle = \langle x|x\rangle - \langle x|y\rangle - \langle y|x\rangle + \langle y|y\rangle = 0$, so $x = y$.

Orthogonal bases and morphisms

In **FHilb**, morphism between two commutative dagger Frobenius structures acts as function on copyable states if and only if it is comonoid homomorphism.

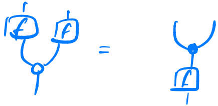
A function

$f : \{d_1, \ldots, d_n\} \longrightarrow \{e_1, \ldots, e_n\}$

induces a linear function

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n$$
$$|d_i\rangle \longmapsto |f(d_i)\rangle$$

that is in fact a comonoid homomorphism



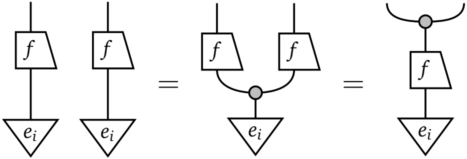
where  $: |d_i\rangle \longmapsto |d_i\rangle \otimes |d_i\rangle$

 $: |e_i\rangle \longmapsto |e_i\rangle \otimes |e_i\rangle$

Orthogonal bases and morphisms

In **FHilb**, morphism between two commutative dagger Frobenius structures acts as function on copyable states if and only if it is comonoid homomorphism.

Proof. Suffices to see about basis of copyable states $\{e_i\}$.



Hence $f(e_i)$ copyable.

Classification in **Rel**

Theorem: Special dagger Frobenius structures in **Rel** correspond exactly to groupoids.

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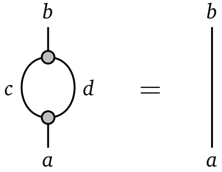
M is single-valued: by speciality $a(M \circ M^{\dagger})b$ iff $a = b$:

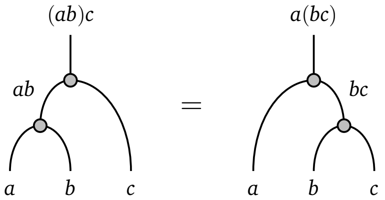


So: if $(c, d)Ma$ and $(c, d)Mb$, must have $a = b$.
May simply write ab for unique c with $(a, b)Mc$.
Remember: ab not always defined!

Classification in **Rel**

Proof. (continued)

Associativity:

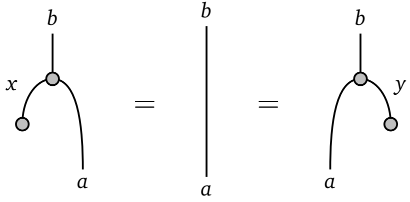


So ab and $(ab)c$ defined exactly when bc and $a(bc)$ are defined, and then $(ab)c = a(bc)$.

Classification in **Rel**

Proof. (continued)

Unitality: for units $x, y \in U$



So: a, b allow $x \in U$ with $xa = b$ iff $a = b$.
And: a, b allow $y \in U$ with $ay = b$ iff $a = b$.
If $z \in U$ then $xz = x$ for some $x \in U$. But then $x = z$!
Units idempotent; multiplication of different ones undefined.

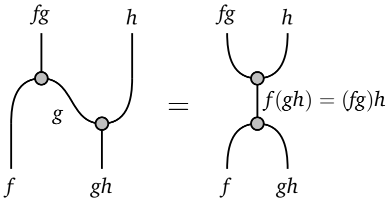
If $xa = a = x'a$, then $a = xa = x(x'a) = (xx')a$, so $x = x'$.
So every element has unique left/right identity.

Classification in **Rel**

Proof. (continued)

Category: U set of objects, A set of morphisms.

If fg defined and gh defined, want $(fg)h = f(gh)$ defined too:

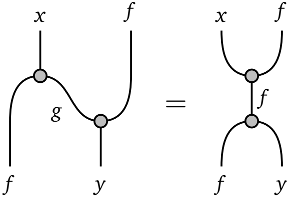


If fg and gh defined then LHS defined, so RHS defined too.

Classification in **Rel**

Proof. (continued)

Inverses: for $f \in A$ with left unit x and right unit y:



□

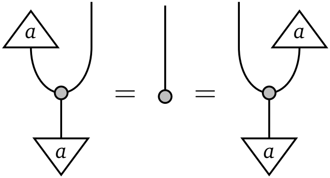
$gf = x = id_{dom(f)}$

$fg = y = id_{cod(f)}$

$\Big\} \Rightarrow g = f^{-1}$

Phases

Let (A, , ) be Frobenius structure in a monoidal dagger category.
State $I \xrightarrow{a} A$ is called phase when:



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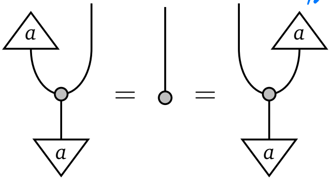
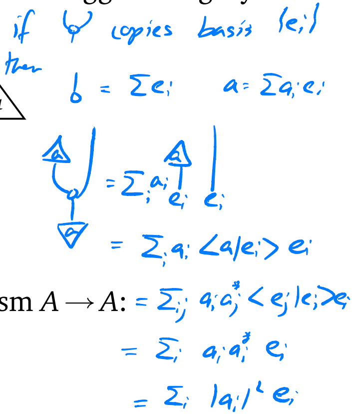
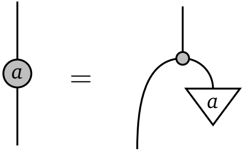




Its (right) phase shift is the following morphism $A \to A$:



Example phases

- For classical structure in **FHilb** copying basis $\{e_i\}$, vector $a = a_1 e_1 + \cdots a_n e_n$ is phase iff each a_i on unit circle: $|a_i|^2 = 1$.

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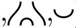
- In a compact dagger category, phases for pair of pants $(A^* \otimes A,$$)$ correspond to unitary morphisms.

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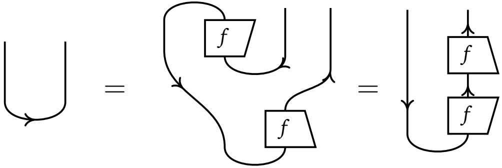
  Proof. The name of an morphism $A \xrightarrow{f} A$ is a phase when:



  But this means $f \circ f^\dagger = \mathrm{id}_A$; similarly $f^\dagger \circ f = \mathrm{id}_A$.

Example phases

- Phases of Frobenius structure $\mathbb{M}_n$ in **FHilb** form set $U(n)$ of n-by-n unitary matrices. Hence phases of $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ range over $U(k_1) \times \cdots \times U(k_n)$.

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- The phases of a Frobenius structure in **Rel** induced by a group G are elements of that group G itself.

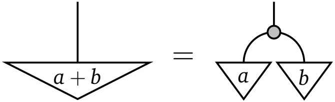
  Proof. For a subset $a \subseteq G$, equation defining phases reads

  $$\{g^{-1}h \mid g, h \in a\} = \{1_G\} = \{hg^{-1} \mid g, h \in a\}.$$

  So if $g \in G$, then $a = \{g\}$ is a phase. But if a contains distinct elements $g \neq h$ of G, cannot be phase. Similarly, $a = \emptyset$ not phase. Hence a phase precisely when singleton $\{g\}$.

Phase group

In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit  and:



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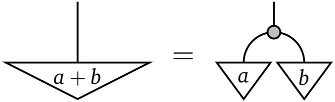
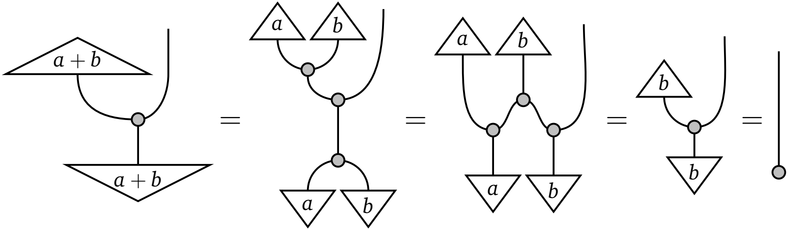


Proof. This is again a well-defined phase:



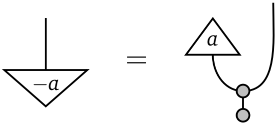
The flipped equation follows similarly.

Associativity is clear, hence phases form a monoid.

Phase group

Proof. (continued)

Left-inverse of phase a is:



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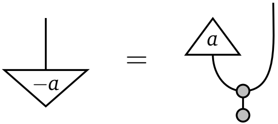

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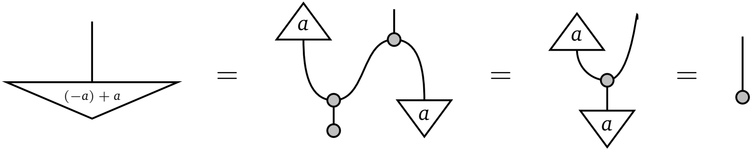

Phase group

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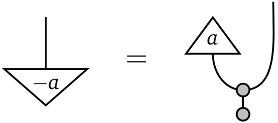

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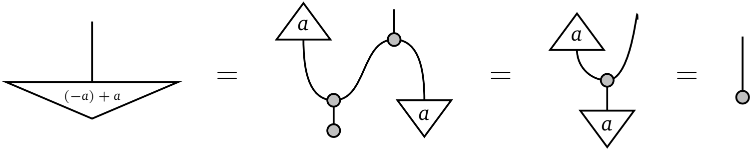

Similarly there is right-inverse. But in monoids, left and right inverses are equal: $l = l(xr) = (lx)r = r$. □

Example phase groups

- In **FHilb**, the phase group for the pair of pants Frobenius structure is the unitary group.

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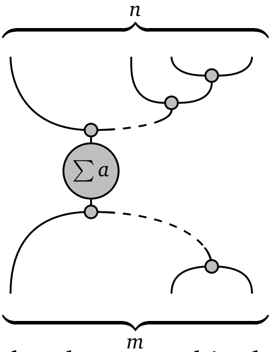
  In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.
- In **Rel**, the phase group induced by a group G is the group itself.

Phased spider theorem

Let $(A,$ ⅄, ⫯$)$ be classical structure in braided monoidal dagger category. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built of finitely many ⅄, ⫯, id, σ and phases using $\circ$, $\otimes$, and $\dagger$, equals



where a ranges over all the phases used in the diagram.

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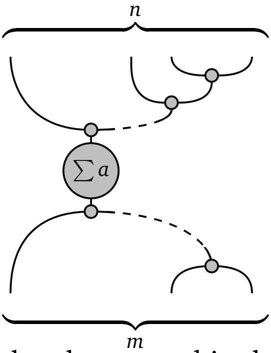


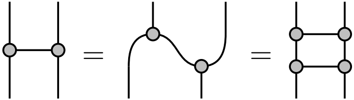
where a ranges over all the phases used in the diagram.

Proof. Use braidings to have all phases dangle at the bottom. Apply Spider Theorem. Use phase addition to reduce to single phase $\sum a$ on bottom right. Apply Spider Theorem again. □

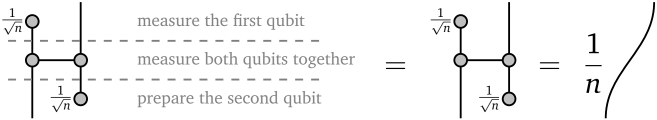
State transfer

State transfer protocol: transfer state of Hilbert space H from one system to another, with success probability $1/\dim(H)^2$.

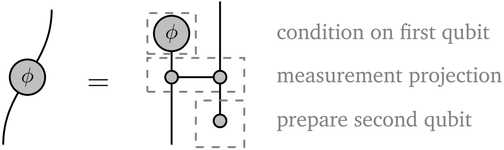
May be lax in drawing, e.g. projection $H \otimes H \to H \otimes H$:



The procedure looks like this:



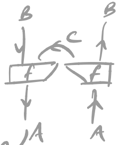
Extra challenge: apply phase gate while transferring state



Problem:

is not *really* a measurement because it uses pure states $|\psi\rangle \in \mathbb{C}^2$ whereas it needs mixed states $\sum_i |\psi_i\rangle\langle\psi_i| : \mathbb{C}^2 \to \mathbb{C}^2$ "density matrices"

because measuring one half of a bipartite pure state is generally a mixed state.



Answer:

don't work in FHilb, but CPM(FHilb).

if $\mathcal{C}$ is dagger compact so is CPM($\mathcal{C}$).

So can use same pictures, but also distinguish "quantum wires" and "classical wires".

a state $I \to H$ in FHilb is a pure state (like $|\psi\rangle$)

CPM(FHilb) — mixed state (like $\sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i|$)

Summary

- Frobenius structures: interacting co/monoid, self-duality
- Normal forms: spider theorems
- Frobenius law: justified by coherence
- Classification: matrix algebras, bases, groupoids
- Phases: unitary operators, state transfer

Next week: interaction between two Frobenius structures