Introduction to Quantum Programming and Semantics
Week 2: Monoidal categories

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Tensor products

Function $f : U \times V \rightarrow W$ is **bilinear** when it is linear in each variable. 

Tensor product of vector spaces $U$ and $V$ is a vector space $U \otimes V$ with bilinear $f : U \times V \rightarrow U \otimes V$ such that for every bilinear $g : U \times V \rightarrow W$ there exists unique linear $h : U \otimes V \rightarrow W$ such that $g = h \circ f$.

Hilbert space with $\langle u \otimes v | u' \otimes v' \rangle = \langle u|u' \rangle \langle v|v' \rangle$.
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![Diagram](image)

Hilbert space with $\langle u \otimes v | u' \otimes v' \rangle = \langle u | u' \rangle \langle v | v' \rangle$

If $H \xrightarrow{f} H'$ and $K \xrightarrow{g} K'$ then $f \otimes g : H \otimes K \rightarrow H' \otimes K'$

$$(f \otimes g) = \begin{pmatrix}
(f_{11}g) & (f_{12}g) & \cdots & (f_{1n}g) \\
(f_{21}g) & (f_{22}g) & \cdots & (f_{2n}g) \\
\vdots & \vdots & \ddots & \vdots \\
(f_{m1}g) & (f_{m2}g) & \cdots & (f_{mn}g)
\end{pmatrix}$$
Monoidal categories

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.
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- physical systems, and physical processes governing them;
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- logical propositions, and implications between them.

Monoidal category theory adds the idea of parallelism:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using separate proofs of $P$ and $Q$ to construct a proof of the conjunction ($P$ and $Q$).
Why so serious?

- Let $A$, $B$ and $C$ be processes, and let $\otimes$ be parallel composition.
- What *relationship* should there be between these systems?

\[
(A \otimes B) \otimes C \quad A \otimes (B \otimes C)
\]

- It’s not right to say they’re *equal*, since even just for sets,

\[
(S \times T) \times U \neq S \times (T \times U).
\]

- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
- How do we treat *trivial* systems?
- What should the relationship be between $A \otimes B$ and $B \otimes A$?
Monoidal category

is a category $\mathbf{C}$ equipped with the following data:

- a tensor product functor

$$\otimes: \mathbf{C} \times \mathbf{C} \to \mathbf{C};$$

- a unit object

$$I \in \text{Ob}(\mathbf{C});$$

- an associator natural isomorphism

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C);$$

- a left unitor natural isomorphism

$$I \otimes A \xrightarrow{\lambda_A} A;$$

- and a right unitor natural isomorphism

$$A \otimes I \xrightarrow{\rho_A} A.$$
Monoidal category
must satisfy triangle and pentagon equations:

\[(A \otimes I) \otimes B \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B)\]
\[
\rho_A \otimes \text{id}_B \quad \text{id}_A \otimes \lambda_B
\]
\[
A \otimes B
\]

\[
(A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)
\]
\[
\alpha_{A,B,C} \otimes \text{id}_D
\]
\[
(A \otimes (B \otimes C)) \otimes D
\]
\[
((A \otimes B) \otimes C) \otimes D
\]
\[
(A \otimes B) \otimes (C \otimes D)
\]
\[
A \otimes (B \otimes (C \otimes D))
\]
\[
A \otimes ((B \otimes C) \otimes D)
\]
\[
\text{id}_A \otimes \alpha_{B,C,D}
\]
\[
\alpha_{A,B,C,D}
\]
\[
\alpha_{A \otimes B,C,D}
\]
\[
\alpha_{A,B,C \otimes D}
\]
Monoidal category
must satisfy triangle and pentagon equations:

\[
\begin{align*}
(A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B) \\
\rho_A \otimes \text{id}_B & \quad \text{id}_A \otimes \lambda_B \\
A \otimes B &
\end{align*}
\]

\[
\begin{align*}
(A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} A \otimes ((B \otimes C) \otimes D) \\
\alpha_{A,B,C} \otimes \text{id}_D & \quad \text{id}_A \otimes \alpha_{B,C,D} \\
((A \otimes B) \otimes C) \otimes D & \\
\alpha_{A\otimes B,C,D} & \quad \alpha_{A,B,C\otimes D} \\
(A \otimes B) \otimes (C \otimes D) & \quad A \otimes (B \otimes (C \otimes D))
\end{align*}
\]

Coherence theorem for monoidal categories: If the pentagon and triangle equations hold, so does any well-typed equation built from \(\alpha, \lambda, \rho\) and their inverses. (to appreciate this, try to prove \(\lambda_I = \rho_I\)!)

Set is monoidal

- tensor product is Cartesian product of sets
- tensor unit is a chosen singleton set \( \{\bullet\} \)
- associators \( (A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C) \)
  defined by \( ((a, b), c) \mapsto (a, (b, c)) \)
- left unitors \( I \times A \xrightarrow{\lambda_A} A \) defined by \( (\bullet, a) \mapsto a \)
- right unitors \( A \times I \xrightarrow{\rho_A} A \) defined by \( (a, \bullet) \mapsto a \)

Other tensor products exist, this one is canonical for classical theory
\[(c, (a, b)) \in T \iff (c, a) \in R \quad \text{and} \quad (c, b) \in S\]

\(\langle N, \leq \rangle\) can be regarded as a category:

- objects are \(\mathbb{N}\)
- there is a morphism \(m \rightarrow n\) exactly when \(m \leq n\)

\(\langle N, \leq \rangle\) can be made monoidal by:

- **Least upper bound:** \(\max(m, n) := \max(m, n)\)

\[
\begin{array}{ccc}
\\downarrow & \downarrow & \downarrow \\
? & \max(m, n) & \\downarrow \\
? & \\swarrow & \\searrow \\
m & m' & m = \max(m, m') \\checkmark \\
n & n' & n = \max(n, n') \\checkmark
\end{array}
\]

- \(I := 0\) \quad \text{if} \quad \max(0, n) = n \quad \checkmark

- **Multiplication:** \(m \cdot n := mn\)

\[
\begin{array}{ccc}
? & \cdot & ? \\
\\swarrow & \downarrow & \\searrow \\
m \cdot m' & m \cdot n' & m \cdot \min(m', n') \\checkmark \\
? & \\uparrow & ? \\
I & I \cdot n & n \\checkmark
\end{array}
\]

- **Addition:**
Set is monoidal

- **tensor product** is Cartesian product of sets
- **tensor unit** is a chosen singleton set \(\{\bullet\}\)
- **associators** \((A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)\)
  defined by \(((a, b), c) \mapsto (a, (b, c))\)
- **left unitors** \(I \times A \xrightarrow{\lambda^A} A\) defined by \((\bullet, a) \mapsto a\)
- **right unitors** \(A \times I \xrightarrow{\rho^A} A\) defined by \((a, \bullet) \mapsto a\)

Other tensor products exist, this one is canonical for classical theory
**Rel** is monoidal

\[ \begin{align*}
(A, B) & \quad A \times B \\
(\mathcal{R}, \mathcal{S}) & \quad \mathcal{R} \subseteq A \times A' \\
(\mathcal{A}', \mathcal{B}') & \quad \mathcal{A}' \times \mathcal{B}'
\end{align*} \]

- **tensor product** is Cartesian product of sets \( \{((a, b), (a', b')) \mid (a, b) \in \mathcal{R} \text{ and } (a', b') \in \mathcal{S} \} \)
on morphisms: \((a, c)(R \times S)(b, d)\) if and only if \(aRb\) and \(cSd\)

- **tensor unit** is a chosen singleton set = \(\{\bullet\}\)

- **associators** \((A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)\) are the relations defined by \(((a, b), c) \sim (a, (b, c))\)

- **left unitors** \(I \times A \xrightarrow{\lambda_A} A\) are the relations defined by \((\bullet, a) \sim a\)

- **right unitors** \(A \times I \xrightarrow{\rho_A} A\) are the relations defined by \((a, \bullet) \sim a\)

This is **not** a categorical product in **Rel**
Hilb is monoidal

- tensor product \( \otimes : \text{Hilb} \times \text{Hilb} \to \text{Hilb} \) is tensor product

- tensor unit \( I \) is the one-dimensional Hilbert space \( \mathbb{C} \)

- associators \( (H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K) \)
  defined by \( (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \)

- left unitors \( \mathbb{C} \otimes H \xrightarrow{\lambda_H} H \) defined by \( 1 \otimes u \mapsto u \)

- right unitors \( H \otimes \mathbb{C} \xrightarrow{\rho_H} H \) defined by \( u \otimes 1 \mapsto u \)

Other tensor products exist, this one is canonical for quantum theory

\[
H \oplus H' = H \times H' \quad \langle (x, y) | (u, v) \rangle_{H \oplus H'} = \langle x | u \rangle_H + \langle y | v \rangle_{H'}
\]
Interchange

Any morphisms $A \overset{f}{\to} B$, $B \overset{g}{\to} C$, $D \overset{h}{\to} E$ and $E \overset{j}{\to} F$ in a monoidal category satisfy the interchange law:

$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$$

Proof:

$$(g \circ f) \otimes (j \circ h) = \otimes((g \circ f), (j \circ h))$$

$$(g \circ f) \otimes (j \circ h) = \otimes((g, j) \circ (f, h)) \quad \text{(composition in } C \times C)$$

$$(g \circ f) \otimes (j \circ h) = ((g \otimes j) \circ (f \otimes h)) \quad \text{(functoriality of } \otimes)$$

$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$$
Graphical calculus

For morphisms $A \overset{f}{\to} B$ and $C \overset{g}{\to} D$, draw $A \otimes C \overset{f \otimes g}{\to} B \otimes D$ as:

```
  B  D
  |   |
  f  g
  A  C
```
Graphical calculus

For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, draw $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ as:

```
  B ---- D
  |     |
  f     g
  |     |
  A ---- C
```

The tensor unit $I$ is drawn as the empty diagram:
Graphical calculus

For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, draw $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ as:

$$
\begin{array}{c}
B \\ f \\
A \\
\end{array}
\quad
\begin{array}{c}
D \\ g \\
C \\
\end{array}
$$

The tensor unit $I$ is drawn as the empty diagram:

Unitors and associators are also not depicted:

$$
\begin{array}{c}
A \\
\lambda_A \\
\end{array}
\quad
\begin{array}{c}
A \\
\rho_A \\
\end{array}
\quad
\begin{array}{c}
A \\
\alpha_{A,B,C} \\
\end{array}
\quad
\begin{array}{c}
B \\
C \\
\end{array}
$$

Coherence is essential for the graphical calculus: as there can only be a single morphism built from their components of any given type, it doesn’t matter that their graphical calculus encodes no information.
Graphical calculus

Interchange law trivialises:

\[(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)\]

Apparent complexity of monoidal categories just complexity of geometry of the plane. In geometrical notation complexity vanishes.
Isotopy

Two diagrams are planar isotopic when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane
- input and output wires terminate at lower and upper boundaries
- components never intersect

(Height of diagrams may change, and input/output wires may slide horizontally along boundary, but may not change order)
Theorem: well-formed equation $f = g$ in monoidal category follows from the axioms $\iff$ it holds graphically up to planar isotopy
Correctness

**Theorem:** well-formed equation $f = g$ in monoidal category follows from the axioms $\iff$ it holds graphically up to planar isotopy

- $P(f, g) =$ ‘under the axioms of a monoidal category, $f = g$’
- $Q(f, g) =$ ‘graphically, $f$ and $g$ are planar isotopic’

**Soundness** is the assertion that $P(f, g) \Rightarrow Q(f, g)$ for all such $f$ and $g$ (easy to prove: just check each axiom)

**Completeness** is the converse: $Q(f, g) \Rightarrow P(f, g)$ for such $f$ and $g$ (harder: must show that planar isotopy is generated by finite set of moves, each being implied by the monoidal axioms)
Cannot ‘look inside’ object to see elements, must use morphisms. A state of an object $A$ is a morphism $I \to A$. 

\[
\begin{array}{c}
A \\
\downarrow \\
a
\end{array}
\]
States

Cannot ‘look inside’ object to see elements, must use morphisms. A state of an object $A$ is a morphism $I \rightarrow A$.

- In $\text{Hilb}$: linear functions $\mathbb{C} \xrightarrow{f} H$, so elements of $H$ $f(2) = 2 \cdot f(1)$
- In $\text{Set}$: functions $\{\bullet\} \rightarrow A$, so elements of $A$
- In $\text{Rel}$: relations $\{\bullet\} \xrightarrow{R} A$, so subsets of $A$
Effects

**An effect** on an object $A$ is a morphism $A \rightarrow I$

Interpret effect as *observation* that a system has some property

States, effects, and other morphisms, build up **histories**:

$$
\begin{array}{c}
\text{e.g. in } \text{Set:} & A \xrightarrow{f} \{0\} \quad \text{only one!} \\
\text{Rel:} & A \leftarrow \{0\} \quad \text{subsets of } A \\
\text{Hilb:} & H \rightarrow C \quad \text{vectors in } H
\end{array}
$$
Joint states

A morphism $I \xrightarrow{c} A \otimes B$ is a joint state of $A$ and $B$.

It is a product state when of the form $I \xrightarrow{\lambda_{I}^{-1}} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:

It is entangled when not a product state.
Joint states: examples

- **In Set:**
  - *joint states* of $A$ and $B$ are elements of $A \times B$
  - *product states* are elements $(a, b) \in A \times B$
  - *entangled states* don’t exist
Joint states: examples

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- **In Rel:**
  - *joint states* of $A$ and $B$ are subsets of $A \times B$
  - *product states* are ‘square’ subsets $V \times W \subseteq A \times B$
  - *entangled states* are subsets not of this form
Joint states: examples

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  - product states are elements $(a, b) \in A \times B$
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- **In Rel:**
  - joint states of $A$ and $B$ are subsets of $A \times B$
  - product states are ‘square’ subsets $V \times W \subseteq A \times B$
  - entangled states are subsets not of this form

- **In Hilb:**
  - joint states of $H$ and $K$ are elements of $H \otimes K$
  - product states are factorizable states
  - entangled states are entangled states in the quantum sense

\[ e.g. \ (\emptyset) \otimes (\cdot) + (\cdot) \otimes (\emptyset) \in \mathbb{C}^{d} \otimes \mathbb{C}^{d} \]
Summary

- Monoidal category: coherent tensor products
- Sound and complete graphical calculus
- States and effects: histories
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