Introduction to Quantum Programming and Semantics

Week 2: Monoidal categories

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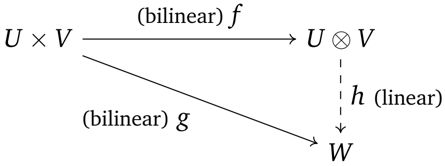
informatics

Tensor products

Function $f\colon U \times V \longrightarrow W$ is bilinear when it is linear in each variable

Tensor product of vector spaces U and V is a vector space $U \otimes V$ with bilinear $f\colon U \times V \longrightarrow U \otimes V$ such that for every bilinear $g\colon U \times V \longrightarrow W$ there exists unique linear $h\colon U \otimes V \longrightarrow W$ such that $g = h \circ f$



Hilbert space with $\langle u \otimes v | u' \otimes v' \rangle = \langle u | u' \rangle \langle v | v' \rangle$

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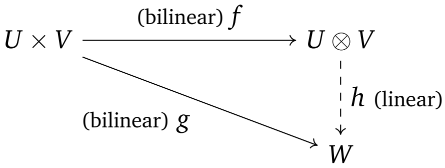

Hilbert space with $\langle u \otimes v | u' \otimes v' \rangle = \langle u | u' \rangle \langle v | v' \rangle$

If $H \xrightarrow{f} H'$ and $K \xrightarrow{g} K'$ then $f \otimes g\colon H \otimes K \to H' \otimes K'$

$$(f \otimes g) = \begin{pmatrix} (f_{11}g) & (f_{12}g) & \cdots & (f_{1n}g) \\ (f_{21}g) & (f_{22}g) & \cdots & (f_{2n}g) \\ \vdots & \vdots & \ddots & \vdots \\ (f_{m1}g) & (f_{m2}g) & \cdots & (f_{mn}g) \end{pmatrix}$$

Monoidal categories

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.

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Monoidal category theory adds the idea of parallelism:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using separate proofs of P and Q to construct a proof of the conjunction (P and Q).

Why so serious?

- Let A, B and C be processes, and let $\otimes$ be parallel composition
- What *relationship* should there be between these systems?

$$(A \otimes B) \otimes C \qquad\qquad A \otimes (B \otimes C)$$

- It's not right to say they're *equal*, since even just for sets,

$$(S \times T) \times U \neq S \times (T \times U).$$

- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
- How do we treat *trivial* systems?
- What should the relationship be between $A \otimes B$ and $B \otimes A$?

Monoidal category

is a category **C** equipped with the following data:

- a tensor product functor

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C};$$

$$\begin{array}{ccc} (A,B) & \longmapsto & A+B \\ \downarrow (f,g) & \longmapsto & \downarrow [f,g] \\ (A',B') & \longmapsto & A'+B' \end{array}$$

- a unit object

$$I \in \mathrm{Ob}(\mathbf{C}); \qquad \emptyset$$

- an associator natural isomorphism

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C);$$

- a left unitor natural isomorphism

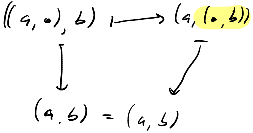
$$I \otimes A \xrightarrow{\lambda_A} A; \qquad \emptyset + A \simeq A$$

- and a right unitor natural isomorphism

$$A \otimes I \xrightarrow{\rho_A} A.$$

Monoidal category

must satisfy triangle and pentagon equations:



$$
\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\
& {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\
& A \otimes B &
\end{array}
$$

$$
\begin{array}{ccc}
\big(A \otimes (B \otimes C)\big) \otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} & A \otimes \big((B \otimes C) \otimes D\big) \\
\uparrow {\scriptstyle \alpha_{A,B,C} \otimes \mathrm{id}_D} & & \downarrow {\scriptstyle \mathrm{id}_A \otimes \alpha_{B,C,D}} \\
\big((A \otimes B) \otimes C\big) \otimes D & & A \otimes \big(B \otimes (C \otimes D)\big) \\
& {\scriptstyle \alpha_{A\otimes B,C,D}} \searrow \quad \nearrow {\scriptstyle \alpha_{A,B,C\otimes D}} & \\
& (A \otimes B) \otimes (C \otimes D) &
\end{array}
$$

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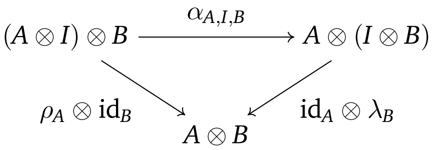
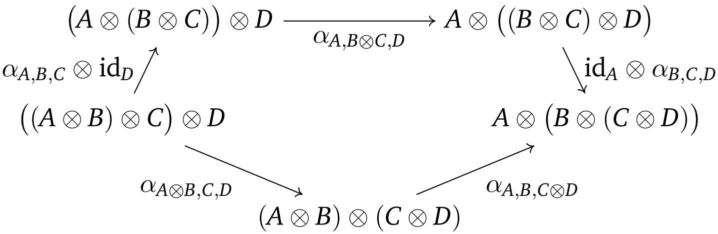
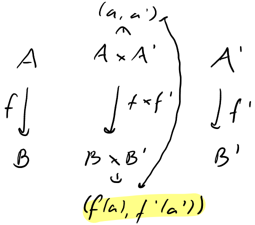


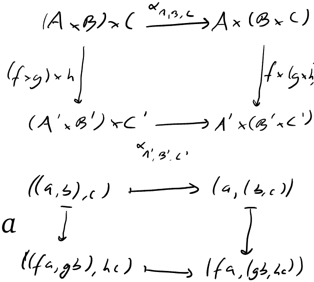


Coherence theorem for monoidal categories: If the pentagon and triangle equations hold, so does any well-typed equation built from α, λ, ρ and their inverses. (to appreciate this, try to prove $\lambda_I = \rho_I$!)

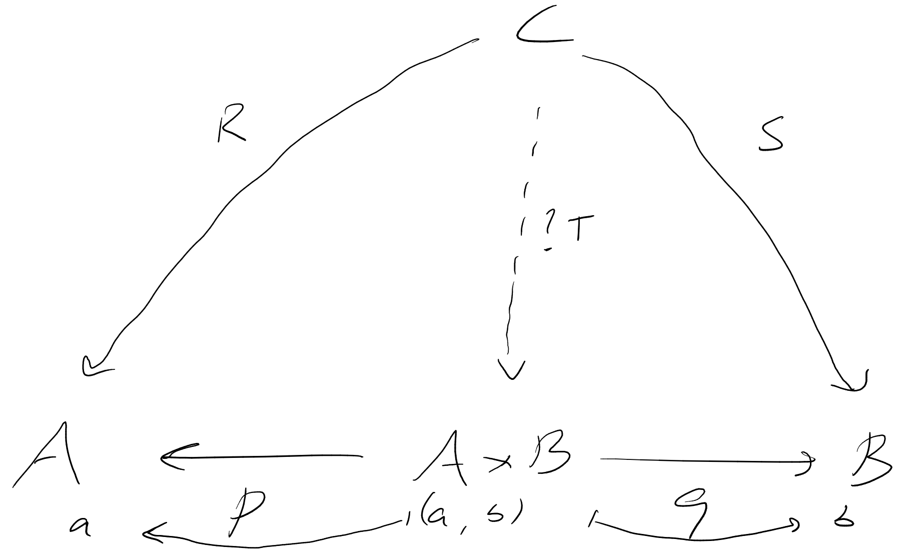
Set is monoidal



- tensor product is Cartesian product of sets
- tensor unit is a chosen singleton set $\{\bullet\}$
- associators $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ defined by $\big((a,b),c\big) \mapsto \big(a,(b,c)\big)$
- left unitors $I \times A \xrightarrow{\lambda_A} A$ defined by $(\bullet, a) \mapsto a$
- right unitors $A \times I \xrightarrow{\rho_A} A$ defined by $(a, \bullet) \mapsto a$



Other tensor products exist, this one is canonical for classical theory



$(c,(a,b)) \in T \iff (c,a) \in R$ and $(c,b) \in S$

$(\mathbb{N}, \leq)$ can be regarded as a category:

objects are $n \in \mathbb{N}$

there is a morphism $m \to n$ exactly when $m \leq n$

it can be made monoidal by:

- least upper bound: $m \otimes n := \max(m,n)$

$$\begin{array}{ccc} m & n & m \otimes n \\ \downarrow & \downarrow & \downarrow ? \\ m' & n' & m' \otimes n' \end{array}$$

$m \leq m', \; n \leq n' \overset{?}{\Longrightarrow} \max(m,n) \leq \max(m',n')$ ✓

$I := 0 \qquad I \otimes n = \max(0,n) = n$ ✓

- multiplication: $m \otimes n := mn$

$m \leq m', \; n \leq n' \overset{?}{\Longrightarrow} mn \leq m'n'$ ✓

$I := 1 \qquad I \otimes n = 1n = n$ ✓

- addition:

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Rel is monoidal

$(A, B) \xrightarrow{(R,S)} (A', B')$ $\qquad$ $A \times B \overset{?}{\dashrightarrow} A' \times B'$

$R \subseteq A \times A'$

$S \subseteq B \times B'$

$? \subseteq (A \times B) \times (A' \times B')$

- tensor product is Cartesian product of sets $\{((a,b),(a',b')) \mid (a,a') \in R \text{ and } (b,b') \in S\}$
  on morphisms: $(a, c)(R \times S)(b, d)$ if and only if aRb and cSd

- tensor unit is a chosen singleton set $= \{\bullet\}$

- associators $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the relations defined by $\big((a, b), c\big) \sim \big(a, (b, c)\big)$

- left unitors $I \times A \xrightarrow{\lambda_A} A$ are the relations defined by $(\bullet, a) \sim a$

- right unitors $A \times I \xrightarrow{\rho_A} A$ are the relations defined by $(a, \bullet) \sim a$

This is not a categorical product in **Rel**

Hilb is monoidal

$H \xrightarrow{f} K$, $H' \xrightarrow{f'} K'$

$H \otimes H' = \mathrm{span}\{h \otimes h' \mid h \in H, h' \in H'\}$

$f \otimes f' : H \otimes H' \to K \otimes K'$, $f \otimes f'(h \otimes h') = f(h) \otimes f'(h')$

- tensor product $\otimes$: **Hilb** $\times$ **Hilb** $\to$ **Hilb** is tensor product

- tensor unit I is the one-dimensional Hilbert space $\mathbb{C}$

- associators $(H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K)$
  defined by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$

- left unitors $\mathbb{C} \otimes H \xrightarrow{\lambda_H} H$ defined by $1 \otimes u \mapsto u$

- right unitors $H \otimes \mathbb{C} \xrightarrow{\rho_H} H$ defined by $u \otimes 1 \mapsto u$

Other tensor products exist, this one is canonical for quantum theory

$H \oplus H' = H \times H'$ $\quad \langle (x,y) \mid (u,v) \rangle_{H \oplus H'} = \langle x \mid u \rangle_H + \langle y \mid v \rangle_{H'}$

Interchange

Any morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $D \xrightarrow{h} E$ and $E \xrightarrow{j} F$ in a monoidal category satisfy the interchange law:

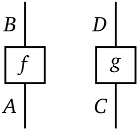
$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$$

Proof:

$$\begin{aligned}
(g \circ f) \otimes (j \circ h) &= \otimes(g \circ f, j \circ h) \\
&= \otimes\big((g, j) \circ (f, h)\big) && (\text{composition in } \mathbf{C} \times \mathbf{C}) \\
&= \big(\otimes(g, j)\big) \circ \big(\otimes(f, h)\big) && (\text{functoriality of } \otimes) \\
&= (g \otimes j) \circ (f \otimes h)
\end{aligned}$$

Graphical calculus

For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, draw $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ as:



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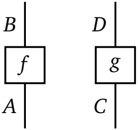

The tensor unit I is drawn as the empty diagram:

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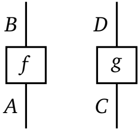


The tensor unit I is drawn as the empty diagram:

Unitors and associators are also not depicted:

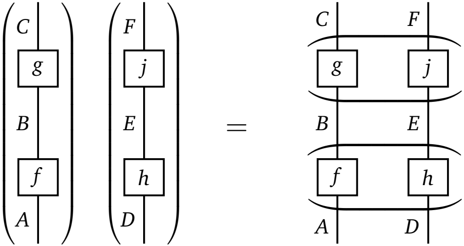
λ_A ρ_A $\alpha_{A,B,C}$

Coherence is essential for the graphical calculus: as there can only be a single morphism built from their components of any given type, it *doesn't matter* that their graphical calculus encodes no information

Graphical calculus

Interchange law trivialises:

$$(g \circ f) \otimes (j \circ h) \quad = \quad (g \otimes j) \circ (f \otimes h)$$

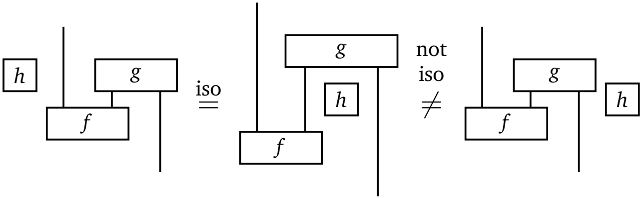


Apparent complexity of monoidal categories just complexity of *geometry of the plane*. In geometrical notation complexity vanishes.

Isotopy

Two diagrams are planar isotopic when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane
- input and output wires terminate at lower and upper boundaries
- components never intersect



(Height of diagrams may change, and input/output wires may slide horizontally along boundary, but may not change order)

Correctness

Theorem: well-formed equation $f = g$ in monoidal category follows from the axioms $\iff$ it holds graphically up to planar isotopy

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Theorem: well-formed equation $f = g$ in monoidal category follows from the axioms $\iff$ it holds graphically up to planar isotopy

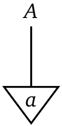
- $P(f,g)$ = 'under the axioms of a monoidal category, $f = g$'
- $Q(f,g)$ = 'graphically, f and g are planar isotopic'

Soundness is the assertion that $P(f,g) \Rightarrow Q(f,g)$ for all such f and g
(easy to prove: just check each axiom)

Completeness is the converse: $Q(f,g) \Rightarrow P(f,g)$ for such f and g
(harder: must show that planar isotopy is generated by finite set of moves, each being implied by the monoidal axioms)

States

Cannot ‘look inside’ object to see elements, must use morphisms.
A state of an object A is a morphism $I \longrightarrow A$.



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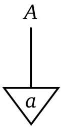


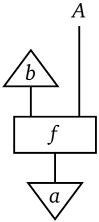
- In **Hilb**: linear functions $\mathbb{C} \xrightarrow{f} H$, so elements of H $f(2) = 2 \cdot f(1)$
- In **Set**: functions $\{\bullet\} \to A$, so elements of A
- In **Rel**: relations $\{\bullet\} \xrightarrow{R} A$, so subsets of A

Effects

e.g. in **Set**: $A \xrightarrow{f} \{\bullet\}$ only one!
Rel: $A \xrightarrow{R} \{\bullet\}$ subsets of A
Hilb: $H \longrightarrow \mathbb{C}$ vectors in H

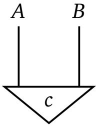
An effect on an object A is a morphism $A \to I$

Interpret effect as *observation* that a system has some property
States, effects, and other morphisms, build up histories:

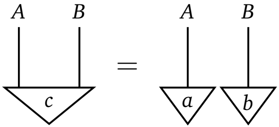


Joint states

A morphism $I \xrightarrow{c} A \otimes B$ is a joint state of A and B.



It is a product state when of the form $I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:



It is entangled when not a product state.

Joint states: examples

- In **Set**:
  - *joint states* of A and B are elements of $A \times B$
  - *product states* are elements $(a, b) \in A \times B$
  - *entangled states* don't exist

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- In **Rel**:
  - *joint states* of A and B are subsets of $A \times B$
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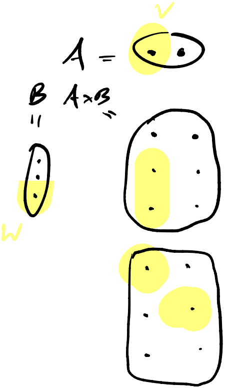
  - *entangled states* are subsets not of this form



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- In **Hilb**:
  - *joint states* of H and K are elements of $H \otimes K$
  - *product states* are factorizable states
  - *entangled states* are entangled states in the quantum sense

$x \in H$

$y \in K$

$x \otimes y \in H \otimes K$

e.g. $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2$

Summary

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- Sound and complete graphical calculus
- States and effects: histories

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