Introduction to Quantum Programming and Semantics

Week 6: Monoids and comonoids

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informatics

Overview

- Monoids: multiplication of states
- Comonoids: ‘copying’ of states
- Cloning: prove no-cloning and no-deleting
- Products: characterize when tensor product is product

Copying

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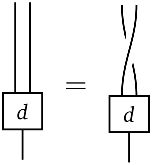

cocommutativity

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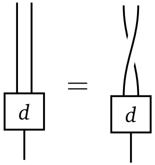

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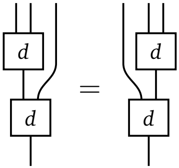

coassociativity

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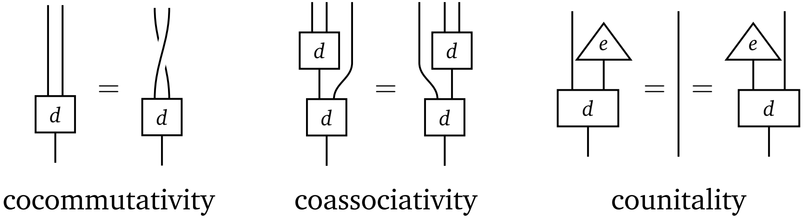

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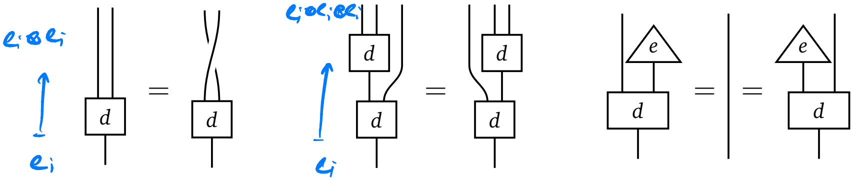

cocommutativity   coassociativity   counitality

Triple (A, d, e) is called (cocommutative) comonoid.

Example comonoids

- In **Set**, the tensor product is a Cartesian product.
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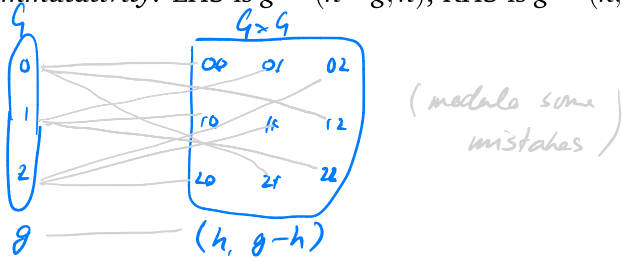
  Counitality: LHS is $g \sim h$ where $h^{-1}g = 1$, RHS is $g \sim 1^{-1}g$.
  The comonoid is cocommutative iff the group is abelian.
  Cocommutativity: LHS is $g \sim (h^{-1}g, h)$, RHS is $g \sim (k, k^{-1}g)$.

$(G,\cdot,1)=(\mathbb{Z}_3,+,0)$



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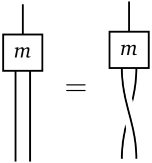
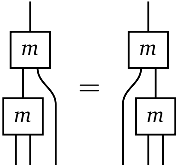
- In **FHilb**, basis $\{e_i\}$ for a Hilbert space gives a cocommutative comonoid, with comultiplication $e_i \mapsto e_i \otimes e_i$ and counit $e_i \mapsto 1$.

Monoids

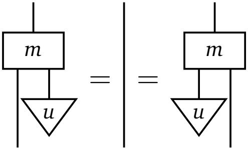
Dually:



commutativity



associativity



unitality

Monoids

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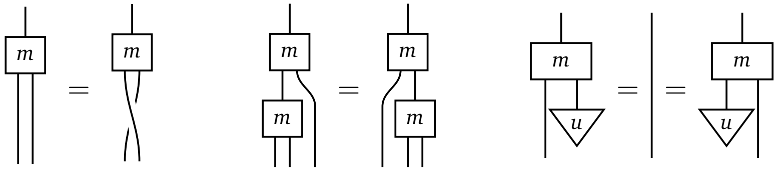

commutativity *associativity* *unitality*

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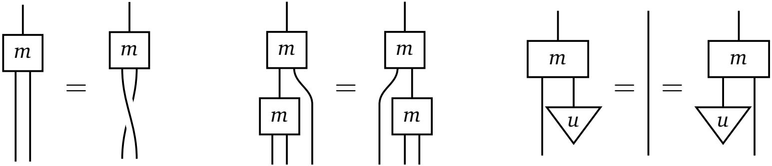

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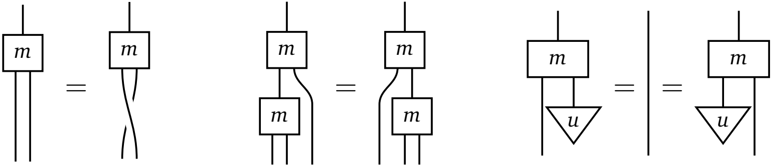

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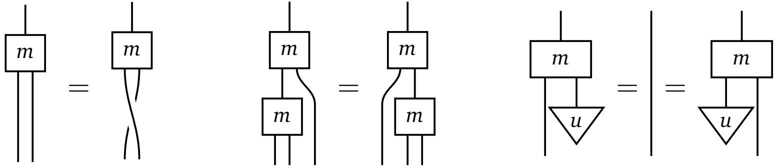

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Triple (A, m, u) is (commutative) monoid. Examples:

- Tensor unit I, with multiplication $\rho_I = \lambda_I$ and unit id_I.
- A monoid in **Set** is just an ordinary monoid; e.g. any group.
- A monoid in **Vect** is an *algebra*: a set where we can add vectors and multiply with scalars, and also multiply vectors bilinearly. E.g. $\mathbb{C}^n$ under pointwise multiplication and unit $(1, 1, \dots, 1)$. E.g. vector space of n-by-n matrices with matrix multiplication.

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Draw comultiplication as , counit as , multiplication as , unit as .

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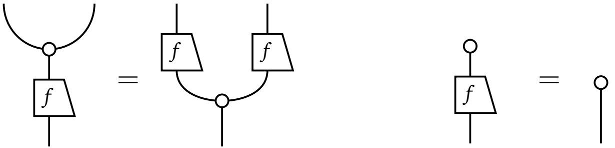

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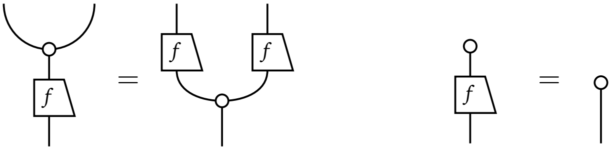

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Draw comultiplication as ⟨comultiplication symbol⟩, counit as ⟨counit symbol⟩, multiplication as ⟨multiplication symbol⟩, unit as ⟨unit symbol⟩.

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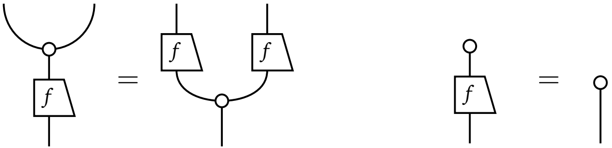

Dually: monoid homomorphism.

Given monoidal category, can build new category of (co)monoids and homomorphisms.

Example homomorphisms

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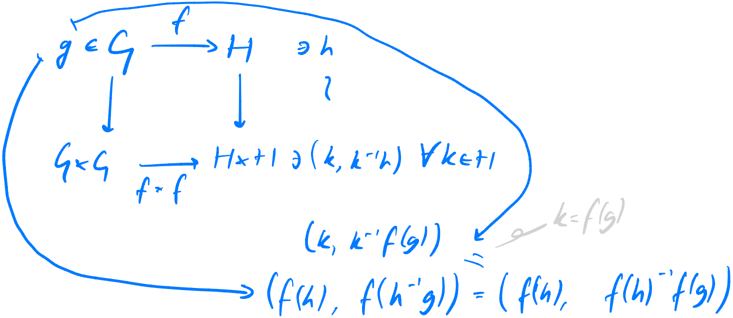

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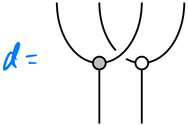
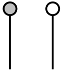
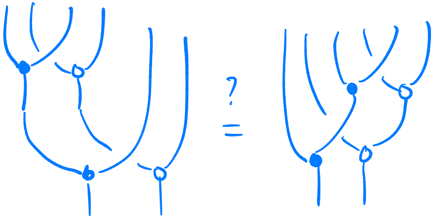
- In **FHilb**, any function $\{d_i\} \xrightarrow{f} \{e_j\}$ between bases extends linearly to a comonoid homomorphism: $d(f(d_i)) = f(d_i) \otimes f(d_i)$ and $e(f(d_j)) = 1 = e(d_j)$.

Product of monoids

Can combine two (co)monoids to single one using braiding:







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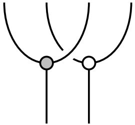

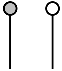

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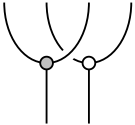

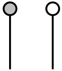

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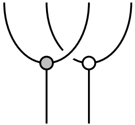

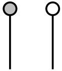

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- In **FHilb**, the product of comonoids on H and K that copy bases $\{d_i\}$ and $\{e_j\}$ is the comonoid copying basis $\{d_i \otimes e_j\}$ of $H \otimes K$.

Dagger

Monoidal dagger category has duality between monoids and comonoids: (A, d, e) is a comonoid if and only if $(A, d^\dagger, e^\dagger)$ is a monoid.

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Example:

- In **Rel**: comultiplication $g \sim (h, h^{-1}g)$ for group G turns into multiplication $(g, h) \sim gh$.

Closure

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Can handle this using names and conames. E.g.:

$$\mathbf{FHilb}(H,K) = \{H \xrightarrow{f} K \mid f \text{ linear}\}$$

is vector space with pointwise operations $(f+g)(x) = f(x) + g(x)$,
Hilbert space with *trace inner product* $\langle f | g \rangle = \mathrm{Tr}(f^\dagger \circ g)$.

To transform morphisms, encode them as vectors in function spaces.

Matrices

One of most important features of matrices: they can be multiplied.
In other words, linear maps $\mathbb{C}^n \to \mathbb{C}^n$ can be composed.
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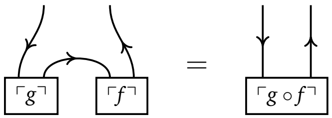
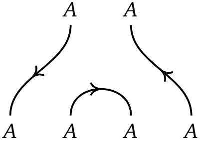
More generally, if an object A in a monoidal category has a dual A^*, then operators $A \xrightarrow{f} A$ correspond bijectively to states $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes A$.
Composition $A \xrightarrow{g \circ f} A$ of operators transfers to states $I \xrightarrow{\ulcorner g \circ f \urcorner} A^* \otimes A$:

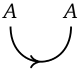


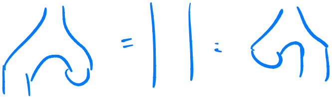
So $A^* \otimes A$ canonically becomes monoid.

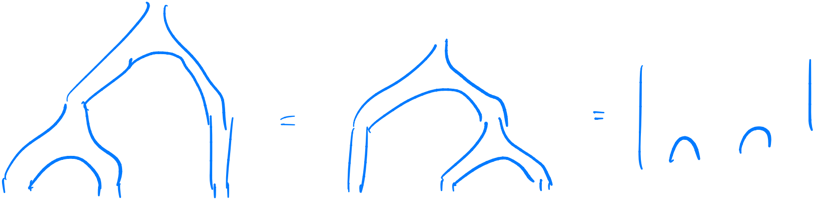
Pair of pants

If $A \dashv A^*$ in monoidal category, then $A^* \otimes A$ is a monoid:









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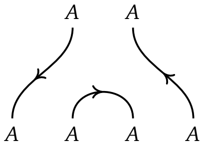
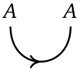
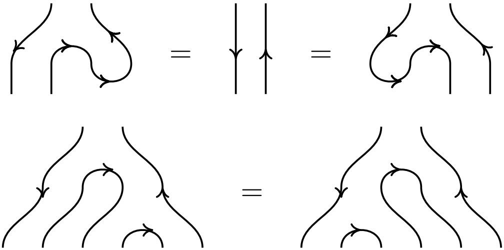




Proof.



Matrix algebras

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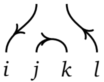
This bijection respects multiplication:



$$= \begin{bmatrix} \langle i| \otimes |l\rangle & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix} \longmapsto \begin{bmatrix} e_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix} = e_{ij} e_{kl}$$

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Closure: instead of injective homomorphism $M \xrightarrow{R} \mathbf{Set}(M, M)$,
consider relation $M \to M^* \times M$ (latter with pair of pants).

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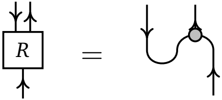
Embedding $R\colon G \to \mathrm{Sym}(G)$ is *regular representation* $g \mapsto R_g$.

$$R_g(h) = g \cdot h$$

Already works for monoids: any M is submonoid of $\mathbf{Set}(M, M)$.
Closure: instead of injective homomorphism $M \xrightarrow{R} \mathbf{Set}(M, M)$,
consider relation $M \to M^* \times M$ (latter with pair of pants).

Abstract embedding of (M, , ) into $M \dashv M^*$:



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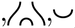
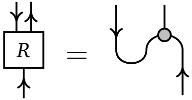
Any monoid $(A,$  $)$ in a monoidal category with $A \dashv A^*$ has monoid homomorphism to $(A^* \otimes A,$  $)$ with right inverse.



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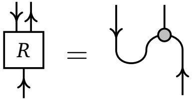
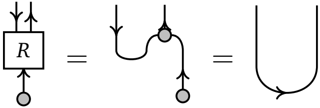


Proof. *R* preserves units:



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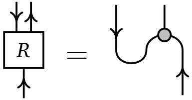

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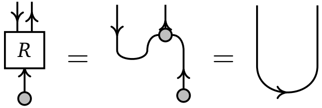
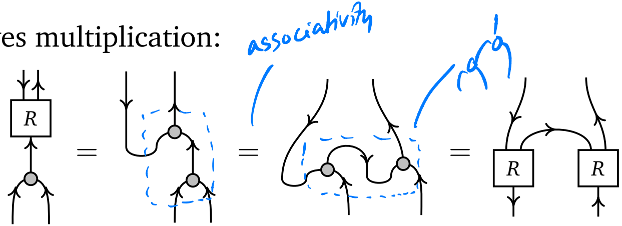


R preserves multiplication:



Finally, *R* has a right inverse ⚲. □

Uniform deleting

Counit $A \xrightarrow{e} I$ tells us we can ‘delete’ A if we want to.
What does it mean to have deletion *systematically* on every object?

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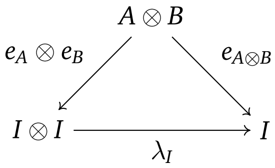

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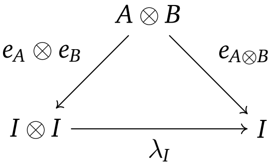

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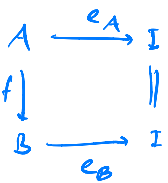

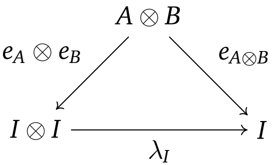

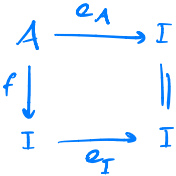

Uniform deleting possible if and only if I is terminal.

Proof. Uniform deleting gives a morphism $A \xrightarrow{e_A} I$ for each object A.
Naturality and $e_I = \mathrm{id}_I$ then show any morphism $A \xrightarrow{f} I$ equals e_A.
Conversely, if I is terminal, choose $e_A : A \to I$ uniquely. □

No-deleting theorem

A preorder is a category that has at most one morphism $A \longrightarrow B$ for any pair of objects A, B.

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Theorem: if a monoidal category with duals has uniform deleting, then it is a preorder.

Proof. Let $A \xrightarrow{f,g} B$ be morphisms. Naturality of e gives:

$$\begin{array}{ccc} A \otimes B^* & \xrightarrow{\quad e_{A\otimes B^*} \quad} & I \\ {\scriptstyle \llcorner f \lrcorner}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_I} \\ I & \xrightarrow{\quad e_I = \mathrm{id}_I \quad} & I \end{array}$$

So $\llcorner f \lrcorner = e_{A\otimes B^*}$, and similarly $\llcorner g \lrcorner = e_{A\otimes B^*}$. Hence $f = g$. □

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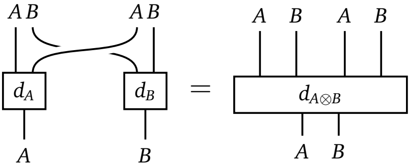
A braided monoidal category has uniform copying if there is a natural transformation $A \xrightarrow{d_A} A \otimes A$ with $d_I = \rho_I$, satisfying cocommutativity and coassociativity, and:



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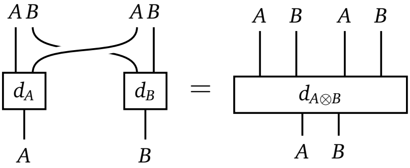
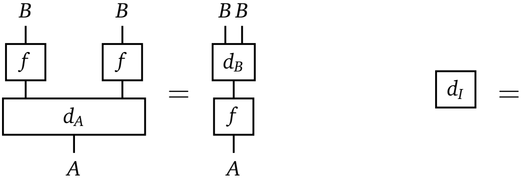


Naturality and $d_I = \rho_I$ look like this for arbitrary $A \xrightarrow{f} B$:



Copying states

Example: **Set** has uniform copying maps $a \mapsto (a, a)$:

$d_1(\bullet) = (\bullet, \bullet) = \rho_1(\bullet)$

both maps $A \times B \to A \times B \times A \times B$ are $(a, b) \mapsto (a, b, a, b)$

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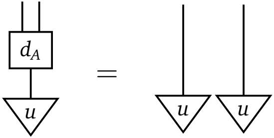
In a braided monoidal category, a state $I \xrightarrow{u} A$ is copyable with respect to a map $A \xrightarrow{d_A} A \otimes A$ when:



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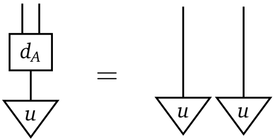

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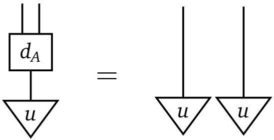
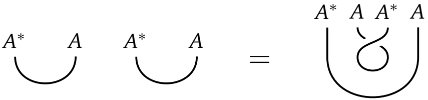


In braided monoidal category with uniform copying, any state is copyable.

Proof. If there is uniform copying, then, by naturality of the copying maps, we have $d_A \circ u = (u \otimes u) \circ \rho_I$ for each state $I \xrightarrow{u} A$. □

Duals vs copying

If a braided monoidal category with duals has uniform copying:



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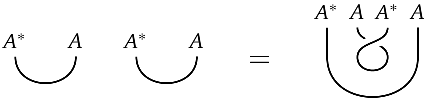

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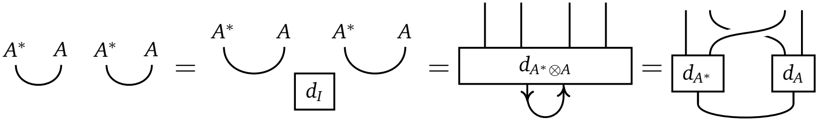

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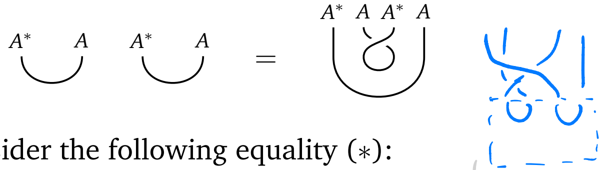

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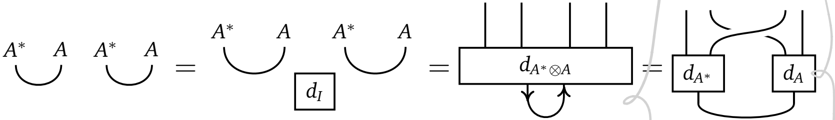
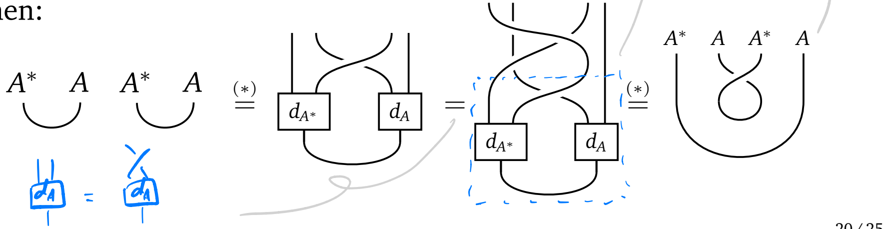
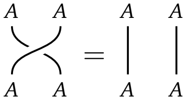


Then:



Duals vs copying

In a braided monoidal category with duals and uniform copying:



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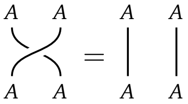
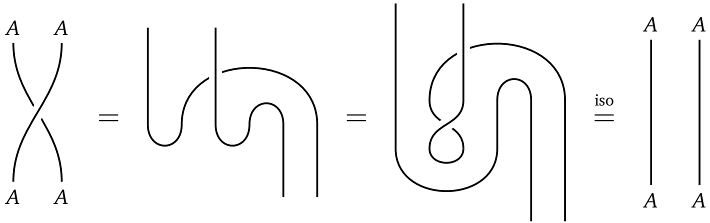
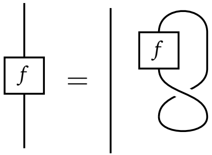


Proof.



No-cloning theorem

If a braided monoidal category with duals has uniform copying, every endomorphism is a multiple of the identity, $f = \mathrm{Tr}(f) \bullet \mathrm{id}$:



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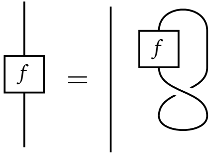

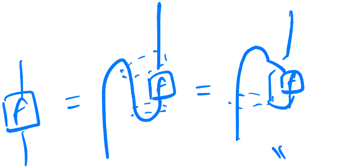

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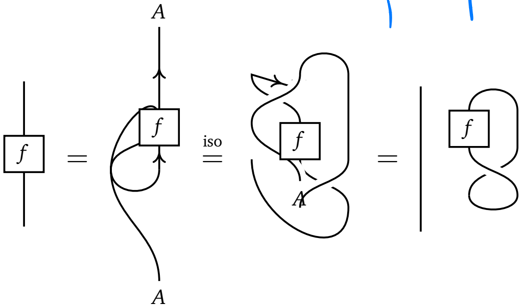

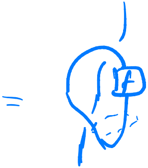

Products

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- tensor products are products and the tensor unit is terminal
- it has uniform copying and deleting, satisfying counitality

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For converse, need to prove $A \otimes B$ is product of A, B.
For $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$, define

$$\begin{pmatrix} f \\ g \end{pmatrix} = (f \otimes g) \circ d$$

$$p_A = \rho_A \circ (\mathrm{id}_A \otimes e_B) \colon A \otimes B \longrightarrow A$$

$$p_B = \lambda_B \circ (e_A \otimes \mathrm{id}_B) \colon A \otimes B \longrightarrow B$$

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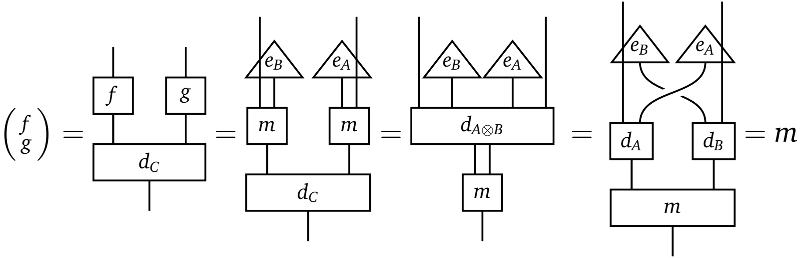

Hence mediating morphisms, if they exist, are unique.

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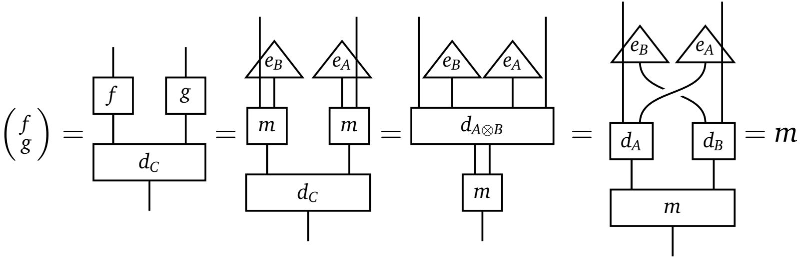

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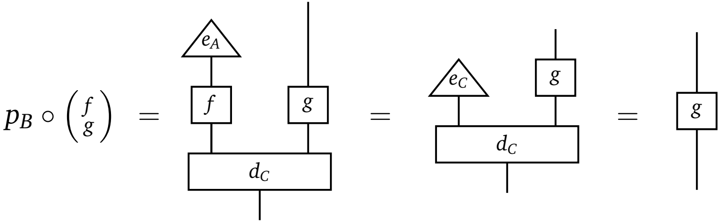
Finally, we show the universal morphism has the right properties:



A similar result holds for g. □

Summary

- Monoids: multiplication on states
- Comonoids: ‘copying’ of states
- Closure: operators form monoids
- Cloning: no-cloning and no-deleting
- Products: characterize when tensor product is product

Next week: interaction between monoids and comonoids