Introduction to Quantum Programming and Semantics

Week 3: Symmetry, scalars, daggers

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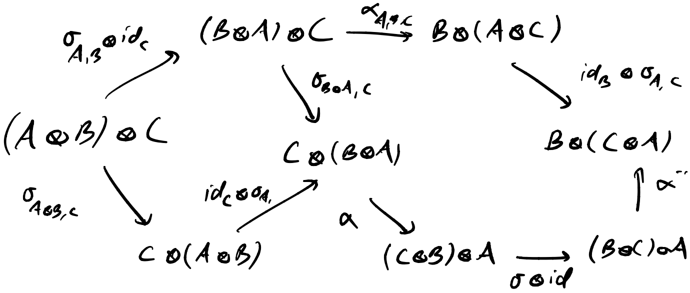
THE UNIVERSITY *of* EDINBURGH informatics

Braiding

A braided monoidal category has a natural isomorphism

$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$



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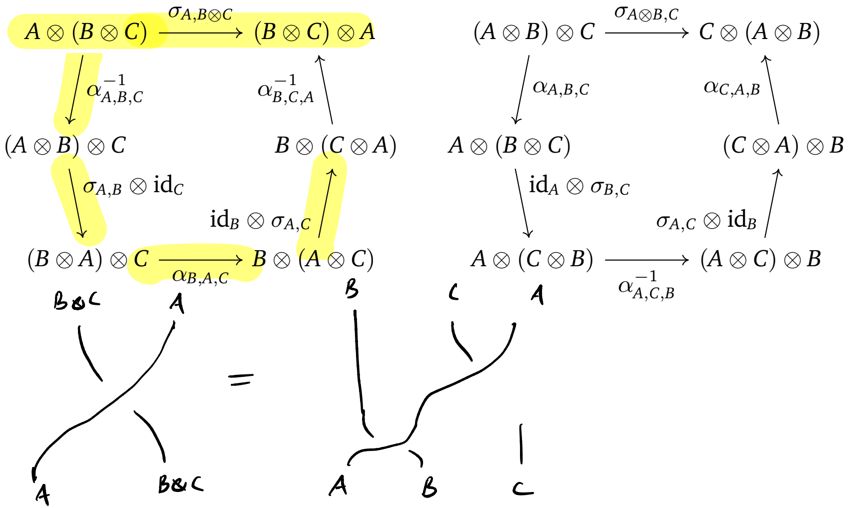

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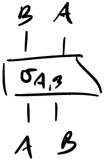
satisfying the hexagon equations

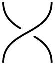
$$\begin{array}{ccc}
A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B\otimes C}} & (B \otimes C) \otimes A \\
\downarrow \alpha^{-1}_{A,B,C} & & \uparrow \alpha^{-1}_{B,C,A} \\
(A \otimes B) \otimes C & & B \otimes (C \otimes A) \\
\downarrow \sigma_{A,B} \otimes \mathrm{id}_C & & \uparrow \mathrm{id}_B \otimes \sigma_{A,C} \\
(B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C)
\end{array}
\qquad
\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\sigma_{A\otimes B,C}} & C \otimes (A \otimes B) \\
\downarrow \alpha_{A,B,C} & & \uparrow \alpha_{C,A,B} \\
A \otimes (B \otimes C) & & (C \otimes A) \otimes B \\
\downarrow \mathrm{id}_A \otimes \sigma_{B,C} & & \uparrow \sigma_{A,C} \otimes \mathrm{id}_B \\
A \otimes (C \otimes B) & \xrightarrow[\alpha^{-1}_{A,C,B}]{} & (A \otimes C) \otimes B
\end{array}$$

- In **Hilb**: $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ defined by $a \otimes b \mapsto b \otimes a$
- In **Set**: $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ defined by $(a, b) \mapsto (b, a)$
- In **Rel**: $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ defined by $(a, b) \sim (b, a)$

Braiding

We draw the braiding as:





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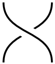

$$B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} A \otimes B$$

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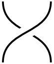
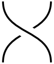
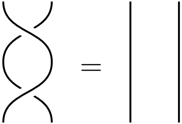
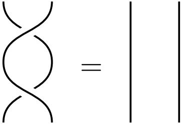




$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \qquad\qquad B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} A \otimes B$$

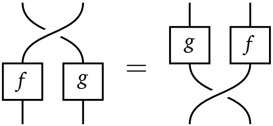
The strands of a braiding cross over each other, so the diagrams are not planar; they are inherently 3-dimensional. Invertibility becomes:

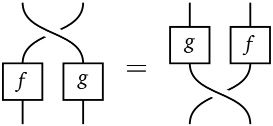




Braiding

Naturality becomes:





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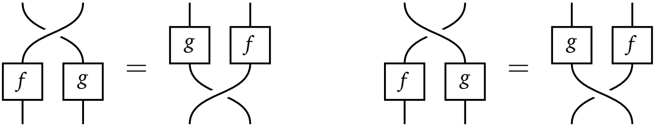
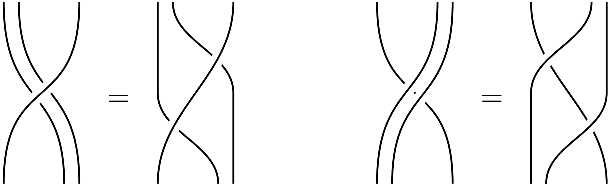
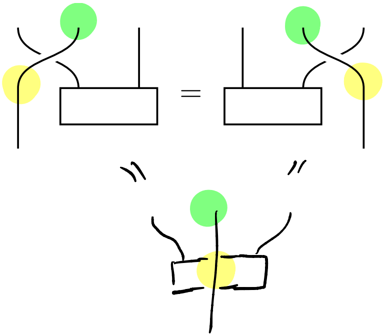


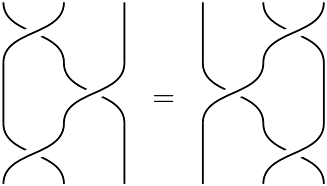
Hexagon equations become:



Graphical calculus

Braided monoidal categories have sound and complete graphical calculus: well-formed equation between morphisms in a braided monoidal category follows from the axioms $\iff$ it holds in the graphical language up to 3-dimensional isotopy.

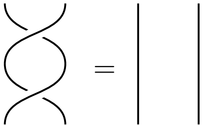




Symmetry

Braided monoidal category is symmetric when

$$\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$$



Strings can pass through each other, no knots: 4d geometry

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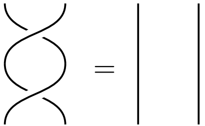
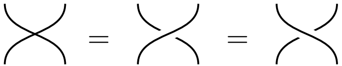


Strings can pass through each other, no knots: 4d geometry

Because $\sigma_{A,B} = \sigma_{B,A}^{-1}$ we may draw



Strictification

- Strictification theorem: every monoidal category is monoidally equivalent to a strict one (unitors and associators are identities)

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- Not every monoidal category is monoidally equivalent to skeletal strict monoidal category

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- Skeletalisation theorem: every category is equivalent to a skeletal one (isomorphic objects are equal)
- Not every monoidal category is monoidally equivalent to skeletal strict monoidal category
- But equivalence $\mathbf{FHilb} \simeq \mathbf{Mat}_{\mathbb{C}}$ is monoidal (tensor product $n \otimes m = nm$, tensor unit 1)

Scalars

Monoidal structure of **Hilb** encodes structure of complex numbers.

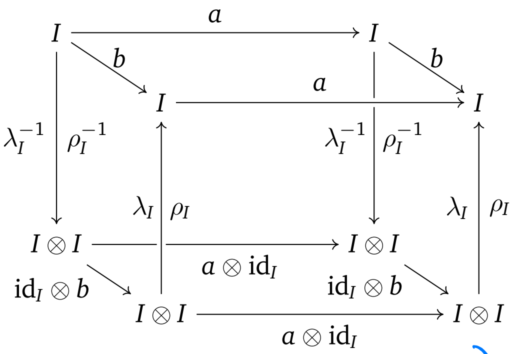
- As a set: $\mathbf{Hilb}(\mathbb{C}, \mathbb{C})$, endomorphisms of tensor unit.
- Multiplication: of complex numbers is given by composition.
- Commutativity: $ab = ba$ for all elements of $\mathbf{Hilb}(\mathbb{C}, \mathbb{C})$.

A scalar in a monoidal category is a morphism $I \to I$.

Can replicate a lot of linear algebra in any monoidal category.

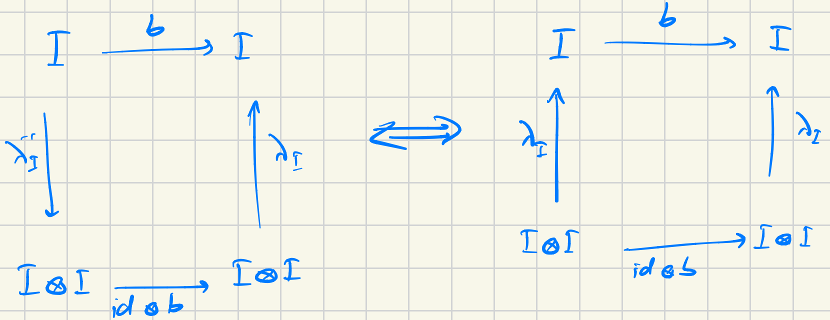
Scalars commute

Lemma: In a monoidal category, scalars commute.
Proof. Consider the following diagram, for any two scalars $I \xrightarrow{a,b} I$:



$\lambda_I = \rho_I : I \otimes I \longrightarrow I$

Side cells: naturality of λ_I and ρ_I. Bottom cell: interchange law.
Vertical arrows: coherence. □



$\Leftarrow$: $\lambda_I \circ (id \otimes b) \circ \lambda_I^{-1}$

$= b \circ \lambda_I \circ \lambda_I^{-1}$

$= b \circ id$

$= b$

$\lambda_I \circ (id \otimes b) = b \circ \lambda_I$

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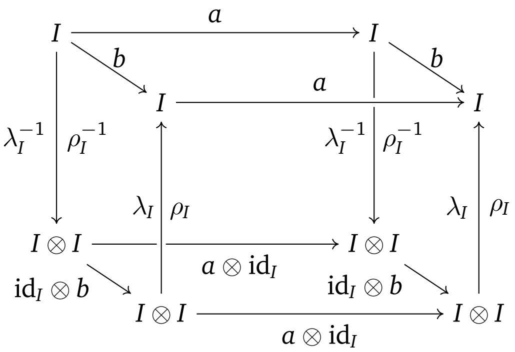

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Graphical calculus

We draw a scalar $I \xrightarrow{a} I$ as a circle:

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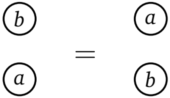
Commutativity of scalars becomes:



Diagrams are isotopic, so it follows from correctness of the graphical calculus that scalars are commutative.

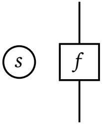
Scalar multiplication

Can multiply linear map $H \xrightarrow{f} J$ with number $c \in \mathbb{C}$, to get $H \xrightarrow{c \cdot f} J$.
Works in any monoidal category.

The left scalar multiplication of morphism $A \xrightarrow{f} B$ with scalar $I \xrightarrow{a} I$ is

$$\begin{array}{ccc} A & \xrightarrow{\quad a \bullet f \quad} & B \\ {\scriptstyle \lambda_A^{-1}} \downarrow & & \uparrow {\scriptstyle \lambda_B} \\ I \otimes A & \xrightarrow{\quad a \otimes f \quad} & I \otimes B \end{array}$$

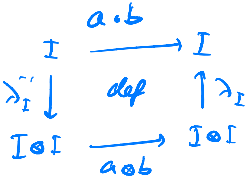
Graphically:



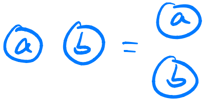
Scalar multiplication

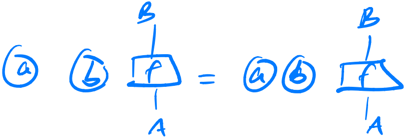
Many familiar properties. For $I \xrightarrow{a,b} I$ and $A \xrightarrow{f} B$, $B \xrightarrow{g} C$:

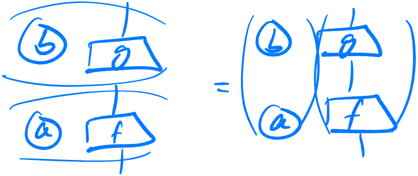
- $\mathrm{id}_I \bullet f = f$
- $a \bullet b = a \circ b$
- $a \bullet (b \bullet f) = (a \bullet b) \bullet f$
- $(b \bullet g) \circ (a \bullet f) = (b \circ a) \bullet (g \circ f)$



Proof. Use graphical calculus. □







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Proof. Use graphical calculus. □

- In **Hilb**: if $a \in \mathbb{C}$ is a scalar and $H \xrightarrow{f} K$ a morphism, then $H \xrightarrow{a \bullet f} K$ is the morphism $v \mapsto af(v)$.
- In **Set**, scalar multiplication is trivial: if $A \xrightarrow{f} B$ is a function, then $\mathrm{id}_1 \bullet f = f$ is again the same function.
- In **Rel**: for any relation $A \xrightarrow{R} B$, $\texttt{true} \bullet R = R$, and $\texttt{false} \bullet R = \emptyset$.

In **Rel**: Scalar is relation $1 \longrightarrow 1$

i.e. $S \subseteq 1 \times 1$

so $S = \emptyset$ or $S = 1 \times 1$

$=:$ false $\qquad =:$ true

AND $=$

$\circ$	f	t
f	f	f
t	f	t

$$f \circ f = \emptyset \circ \emptyset = \{(x,z) \in 1 \times 1 \mid \exists y \in 1 : (x,y) \in \emptyset, (y,z) \in \emptyset\}$$

$$= \emptyset$$

Daggers

In the definition of **FHilb**, something was a bit strange: we didn't use the inner products at all.

Inner products give adjoint linear maps:

$$\langle f^\dagger(x) | y \rangle = \langle x | f(y) \rangle$$

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger \qquad \mathrm{id}_H{}^\dagger = \mathrm{id}_H \qquad (f^\dagger)^\dagger = f$$

Taking adjoints: contravariant involutive functor, identity on objects.

$$\langle \mathrm{id}^\dagger(x) | y \rangle = \langle x | y \rangle$$

$$\langle f^{\dagger\dagger}(x) | y \rangle = \langle x | f^\dagger(y) \rangle, \quad f^{\dagger\dagger} = f$$

$$h = (g \circ f)^\dagger \iff \langle h(x) | y \rangle = \langle x | (g \circ f)(y) \rangle$$
$$= \langle x | g(f(y)) \rangle$$
$$= \langle g^\dagger(x) | f(y) \rangle$$
$$= \langle f^\dagger(g^\dagger(x)) | y \rangle$$

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Conversely, can *recover* inner products from this functor:

$$(\mathbb{C} \xrightarrow{w} H \xrightarrow{v^\dagger} \mathbb{C}) \equiv v^\dagger(w(1)) = \langle 1 | v^\dagger(w(1)) \rangle = \langle v | w \rangle$$

So $\dagger$ and $\langle - | - \rangle$ encode *equivalent* information.

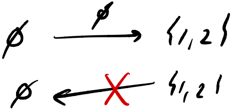
Dagger categories

A dagger on a category $\mathbf{C}$ is an involutive contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$ that is the identity on objects. A dagger category is a category equipped with a dagger.

Examples:

- **Hilb** is a dagger category using adjoint linear maps.
- $\mathbf{Mat}_{\mathbb{C}}$ is a dagger category using the conjugate transpose.
- **Rel** can be given a dagger functor by relational converse: for $S \xrightarrow{R} T$, define $T \xrightarrow{R^\dagger} S$ by setting $t\,R^\dagger\,s$ if and only if $s\,R\,t$.

in Set:



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- **Set** cannot be made into a dagger category: $\mathbf{Set}(A,B)$ has size $|B|^{|A|}$, while $\mathbf{Set}(B,A)$ has size $|A|^{|B|}$.
- **Vect** cannot be given a dagger functor: $\mathbf{Vect}(\mathbb{C},V)$ has a smaller dimension than $\mathbf{Vect}(V,\mathbb{C})$ when V is infinite-dimensional.

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- **Vect** cannot be given a dagger functor: $\mathbf{Vect}(\mathbb{C},V)$ has a smaller dimension than $\mathbf{Vect}(V,\mathbb{C})$ when V is infinite-dimensional.
- **FVect** *can* be given dagger (e.g. by assigning an inner product to objects and constructing adjoints.) But not *canonically* so.

Terminology

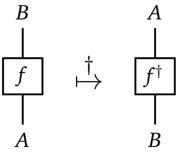
A morphism $A \xrightarrow{f} B$ in a dagger category is:

- the adjoint of $B \xrightarrow{g} A$ when $g = f^\dagger$
- self-adjoint when $f = f^\dagger$
- a projection when $f = f^\dagger$ and $f \circ f = f$
- unitary when both $f^\dagger \circ f = \mathrm{id}_A$ and $f \circ f^\dagger = \mathrm{id}_B$
- an isometry when $f^\dagger \circ f = \mathrm{id}_A$
- a partial isometry when $f^\dagger \circ f$ is a projection
- positive when $f = g^\dagger \circ g$ for some morphism $H \xrightarrow{g} K$

$\langle f(x) | f(y) \rangle = \langle f^\dagger f(x) | y \rangle = \langle x | y \rangle$

Graphical calculus

Depict taking daggers by reflection in horizontal axis.



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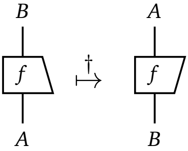
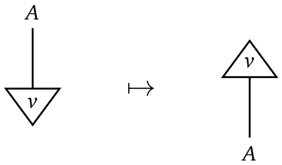


To differentiate, draw morphisms in a way that breaks symmetry.
We also drop the label † from the morphism box.

States, effects, scalars

Dagger gives a correspondence between states and effects:



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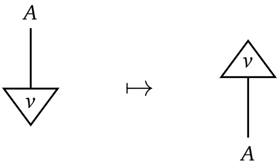
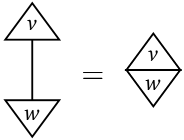


Inner product between two states:

$\langle v|w\rangle =$



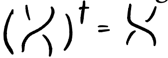
Generalised form of Dirac's bra-ket notation.

Way of the dagger

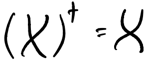
A monoidal dagger category is a dagger category that is also monoidal, such that:

- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms f and g;
- the natural isomorphisms α, λ and ρ are unitary at every stage.

A braided monoidal dagger category is a monoidal dagger category equipped with a unitary braiding.



A symmetric monoidal dagger category is a braided monoidal dagger category for which the braiding is a symmetry.



Summary

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \qquad f^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$f^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad f^\dagger f = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \neq id$$

$$ff^\dagger =$$

$$f^{-1} f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = f f^{-1}$$

- Braiding and symmetry: correct graphical calculus
- Scalars: morphisms $I \rightarrow I$
- Scalars commute
- Scalar multiplication
- Daggers: generalise inner product
- Way of the dagger: monoidal dagger categories