Introduction to Quantum Programming and Semantics: tutorial 1 answers

Exercise 0.1. Composition arises from transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$. This is automatically associative. Identities arise from reflexivity: $x \leq x$. (We don't actually need anti-symmetry, pre-orders also induce categories this way.)

Exercise 0.2. Associativity of the composition of the category is precisely associativity of the monoid multiplication.

Note: pre-orders and monoids are two 'extreme' types of categories. Pre-orders have lots of objects and as few morphisms as possible. Monoids have as few objects as possible and lots of morphisms. In a sense any category is a mixture of these two extremes.

Exercise 0.3. Concatenating paths is associative. Identities arise from paths $v \to v$ of length 0.

Exercise 0.4.

(a) A functor $P \to Q$ by definition consists of a function $f\colon P \to Q$ (on objects) that maps morphisms to morphisms. This means precisely that if $x \leq y$ is a morphism in P, then there must be a morphism $f(x) \leq f(y)$ in Q.

(b) A functor $M \to N$ by definition consists of a function $\{*\} \to \{*\}$ (on objects), and a function $f\colon M \to N$ (on morphisms). The latter has to preserve composition ($f(mn) = f(m)f(n)$) and identities ($f(1) = 1$).

(c) Functors $G \to H$ by definition consist of a function $f\colon \text{Vertices}(G) \to \text{Vertices}(H)$ (on objects), and a function $g\colon \text{Edges}(G) \to \text{Paths}(H)$. The latter induces a function $\text{Paths}(G) \to \text{Paths}(H)$ that respects associativity of composition and identities by definition of composition and identities in the category G.

Exercise 0.5.

(a) Composition of monotone functions is monotone, and the identity is a monotone function.

(b) Composition of homomorphisms is a homomorphism, and the identity is a homomorphism.

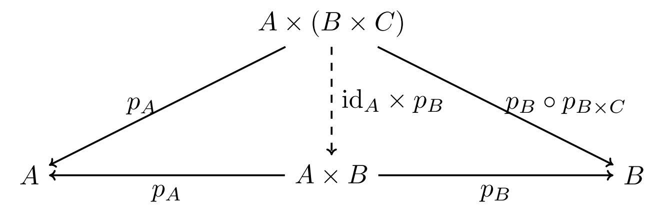
Exercise 0.6.

(a) True: the functor that sends a set A to itself, and a relation $R \subseteq A \times B$ to $\{(b,a) \mid (a,b) \in R\} \subseteq B \times A$, is its own inverse.

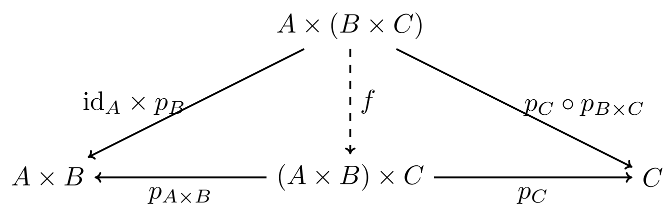
(b) False. Suppose there were an isomorphism $F\colon \mathbf{Set} \to \mathbf{Set}^{\text{op}}$. If a set A has n elements, then $\mathbf{Set}(A, A)$ has n^n elements. Hence also $\mathbf{Set}(F(A), F(A))$ must have n^n elements. Therefore the set $F(A)$ must be finite and have n elements too. Not let B be a finite set with m elements. Then $\mathbf{Set}(A, B) \simeq \mathbf{Set}^{\text{op}}(F(A), F(B)) \simeq \mathbf{Set}(B, A)$, and hence $n^m = m^n$. But e.g. $m = 1$ and $n = 2$ give a contradiction.

(c) True: the assignment on objects that sends $U \in P(X)$ to its complement $X \setminus U \in P(X)$ is functorial, and its own inverse.

Exercise 0.7. The universal property of $A \times B$ provides a morphism that we'll call $\mathrm{id}_A \times p_B$:



The universal property of $(A \times B) \times C$ now provides a morphism $f\colon A \times (B \times C) \longrightarrow (A \times B) \times C$:



Similarly we find a morphism $g\colon (A \times B) \times C \longrightarrow A \times (B \times C)$.
Now $p_A \circ (g \circ f) = p_A \circ \mathrm{id}_{A\times(B\times C)}$ and $p_{B\times C} \circ (g \circ f) = p_{B\times C} \circ \mathrm{id}_{A\times(B\times C)}$. But the universal property of $A \times (B \times C)$ says there is only one morphisms that can satisfy this, so we must have $g \circ f = \mathrm{id}$. Similarly $f \circ g = \mathrm{id}$.