Introduction to Quantum Programming and Semantics: tutorial 1

Exercise 0.1. Let $(P, \leq)$ be a partially ordered set. Show that the following is a category: the objects are the elements x of P, and there is a unique morphism $x \longrightarrow y$ if and only if $x \leq y$.

Exercise 0.2. Let M be a *monoid*: a set M together with an associative binary "multiplication" operation $M \times M \longrightarrow M$ written as $(m, n) \mapsto mn$ and an element $1 \in M$ such that $1m = m = m1$. Show that the following is a category: there is a single object $*$, the morphisms $* \longrightarrow *$ are elements of M, and composition is multiplication. Conversely, show that any category with a single object comes from a monoid in this way.

Exercise 0.3. Let $G = (V, E)$ be a directed graph. Show that the following is a category: objects are vertices $v \in V$, morphisms $v \longrightarrow w$ are paths $v \xrightarrow{e_1} \cdots \xrightarrow{e_n} w$ with $e_i \in E$, and composition is concatenation of paths. Choose $n \geq 5$, and draw a graph with n edges whose category has more than n morphisms.

Exercise 0.4.
(a) If P and Q are partially ordered sets regarded as categories, show that functors $P \longrightarrow Q$ are functions $f \colon P \longrightarrow Q$ that are monotone: if $x \leq y$ then $f(x) \leq f(y)$.
(b) If M and N are monoids regarded as categories, show that functors $M \longrightarrow N$ are functions $f \colon P \longrightarrow Q$ that are homomorphisms: $f(1) = 1$ and $f(mn) = f(m)f(n)$.
(c) If G and H are graphs regarded as categories, what are functors $G \longrightarrow H$?

Exercise 0.5.
(a) Show that partially ordered sets and monotone functions form a category.
(b) Show that monoids and homomorphisms form a category.

Exercise 0.6. Consider the following isomorphisms of categories[1] and determine which hold.
(a) $\mathbf{Rel} \simeq \mathbf{Rel}^{\mathrm{op}}$
(b) $\mathbf{Set} \simeq \mathbf{Set}^{\mathrm{op}}$
(c) For a fixed set X, the powerset $P(X) = \{S \subseteq X\}$ is partially ordered with the subset relation $\subseteq$. Regarding $P(X)$ as a category, $P(X) \simeq P(X)^{\mathrm{op}}$

Exercise 0.7. (Challenge) In any category with binary products, show that $A \times (B \times C) \simeq (A \times B) \times C$.

[1] See Remark 0.6

Definition 0.1 (category)**.** A *category* $\mathbb{C}$ consists of a set of *objects* $\mathrm{Ob}(\mathbb{C})$ and a set of *morphisms* $\mathrm{Mor}(\mathbb{C})$. Each morphism f has its domain object $\mathrm{dom}(f)$ and codomain object $\mathrm{cod}(f)$. A morphism f whose domain is A and codomain is B is written as $f\colon A \to B$.

- For each pair of morphisms f, g such that $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a *composite morphism* $g \circ f$ such that $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$.
- For each object A there is a special morphism called *identity morphism* $\mathrm{id}_A\colon A \to A$.

The composition of morphisms is associative: $h \circ (g \circ f) = (h \circ g) \circ f$, and identity morphisms are the left/right unit of the composition: $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$ for each $f\colon A \to B$.

Definition 0.2 (category)**.** Another equivalent way to define a category is as follows. A *category* $\mathbb{C}$ consists of a set of *objects* $\mathrm{Ob}(\mathbb{C})$ and a set $\mathbb{C}(A, B)$ for each pair of objects A, B. $\mathbb{C}(A, B)$ is the set of morphisms from A to B and called the *hom-set*.

- For each triple of objects A, B, C there is a *composition function* $\circ\colon \mathbb{C}(B, C) \times \mathbb{C}(A, B) \to \mathbb{C}(A, C)$.
- For each object A there is a special morphism called *identity morphism* $\mathrm{id}_A \in \mathbb{C}(A, A)$.

The associativity and unit laws are the same as above.

Definition 0.3 (directed graph)**.** A directed graph consists of a set V of vertices and a set E of edges, together with functions $s, t\colon E \rightrightarrows V$ that assign to each edge e its source $s(e)$ and target $t(e)$. Note that we allow $s(e) = t(e)$, so that there are loops, and we allow multiple edges with the same source and target.

Remark 0.4. In Exercise 0.3, we also allow paths of length 0, which are just vertices.

A category can also be regarded as a directed graph $s = \mathrm{dom}, t = \mathrm{cod}\colon\ \mathrm{Ob}(\mathbb{C}) \rightrightarrows \mathrm{Mor}(\mathbb{C})$ with composition operator.

Definition 0.5 (opposite category)**.** The opposite category $\mathbb{C}^{\mathrm{op}}$ of a category $\mathbb{C}$ has the same objects as $\mathbb{C}$, but the morphisms are reversed: $\mathbb{C}^{\mathrm{op}}(A, B) = \mathbb{C}(B, A)$, and the composition is reversed; if we denote the composition of morphisms in $\mathbb{C}$ by $\circ_{\mathbb{C}}$, then composition $\circ_{\mathbb{C}^{\mathrm{op}}}$ of morphisms in $\mathbb{C}^{\mathrm{op}}$ is defined by $f \circ_{\mathbb{C}^{\mathrm{op}}} g = g \circ_{\mathbb{C}} f$.

For example, if we regard a monoid $(M, \cdot)$ as a category with a single object $*$, then the opposite category M^{op} is the monoid with the same underlying set, but the multiplication $\diamond$ is reversed: $m \diamond n \coloneqq n \cdot m$.

Remark 0.6. Insted of writing the isomorphism of objects as a pair of morphisms $f\colon A \to B$ and $f^{-1}\colon B \to A$, we write $f\colon A \simeq B$. When we omit the description of f and just write $A \simeq B$, it means that there exists an isomorphism $f\colon A \to B$.

In the Exercise 0.4 we are considering the category **Cat** of categories, whose objects are categories and whose morphisms are functors. What is expected to show in this exercise is to construct a functor $F\colon \mathbb{C} \to \mathbb{C}^{\mathrm{op}}$ which has an inverse functor $G\colon \mathbb{C}^{\mathrm{op}} \to \mathbb{C}$, or prove that there is no such functor.