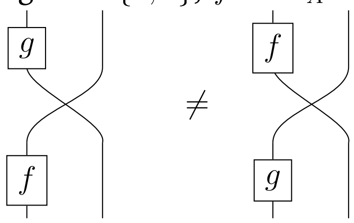
Introduction to Quantum Programming and Semantics: tutorial 2 answers

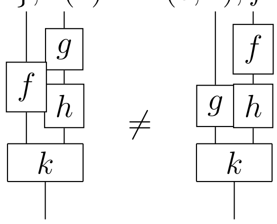
Exercise 1.4.2

Recall that the swap map is natural.

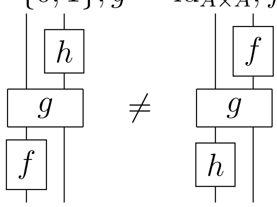
(a) Taking $A = \{0, 1\}$, $f = \mathrm{id}_A$ and $g(0) = 1$ and $g(1) = 0$ in $(\mathbf{Set}, \times)$ shows that



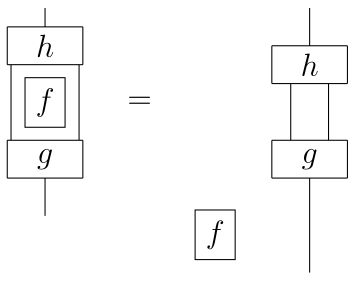
(b) Taking $A = C = \{a\}, B = \{0, 1\}, k(a) = (0, a), f = \mathrm{id}_B$ and $g(0) = 1,\ g(1) = 0$ in $(\mathbf{Set}, \times)$ shows that



(c) Taking $A = \{0, 1\}, g = \mathrm{id}_{A\times A}, f = \mathrm{id}_A$ and $h(0) = 1,\ h(1) = 0$ in $(\mathbf{Set}, \times)$ shows that

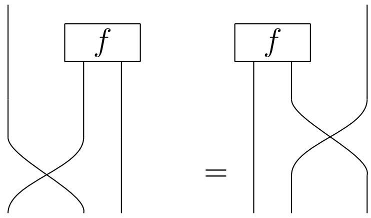


(d) The equation presented below:



is clearly true in the graphical calculus for braided monoidal categories.

(e) The equation presented below:



is clearly true in the graphical calculus for symmetric monoidal categories. Note, that the flipped and braided version of this equation is proved in Exercise 1.4(a).

Exercise 1.4.3

(a) In monoidal categories, equalities of the diagrams hold iff they can be continously deformed into each other using 2-dimensional isotopy. Diagram (1) can be continously deformed into diagram (2), so they are equal. In diagram (3), the scalar k is stuck between the wires of j and h, so it is not equal to (1) and (2).

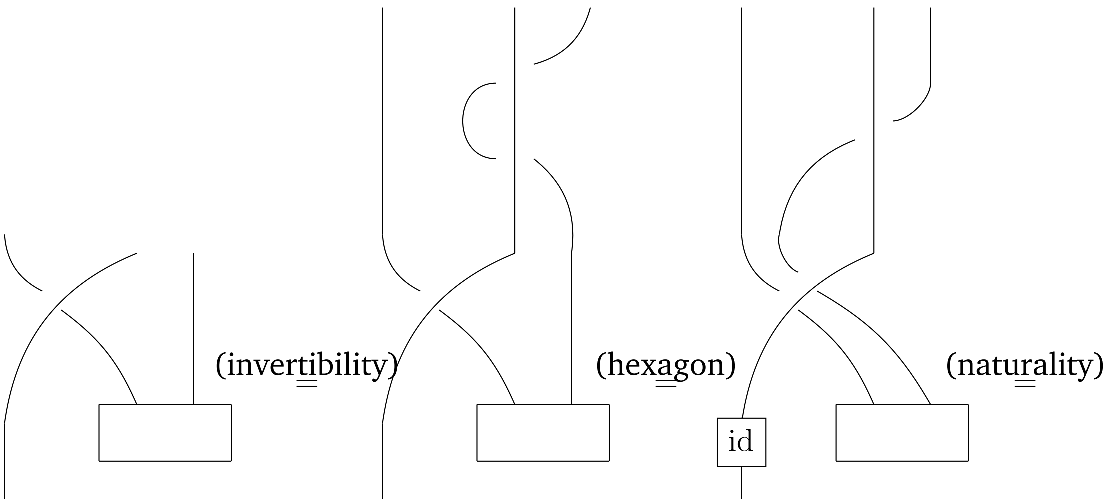
(b) In braided monoidal categories, equalities of the diagrams hold iff they can be continously deformed into each other using 3-dimensional isotopy. From (a) we have $(1) = (2)$. Using 3-dimensional isotopy, we can show $(1) = (2) = (3)$ by taking the scalar k out in the third dimension and then moving it over the enclosing wires.

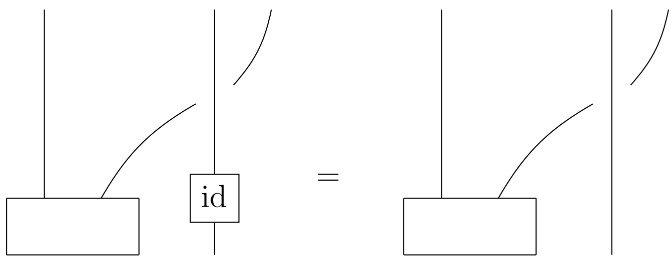
However, we can't show that (4) is equal to the other three diagrams using the axioms of a braided monoidal category. We can still move the scalar k out of the enclosing wires, but we can't uncross the wires themselves. Note, that removing the crossing of the wires in (4) requires that $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$ which is always true for symmetric monoidal categories, but not necessarily for braided monoidal categories.

(c) So, in (c) all diagrams are equal, but in (b) $(1) = (2) = (3) \neq (4)$.

Exercise 1.4.4

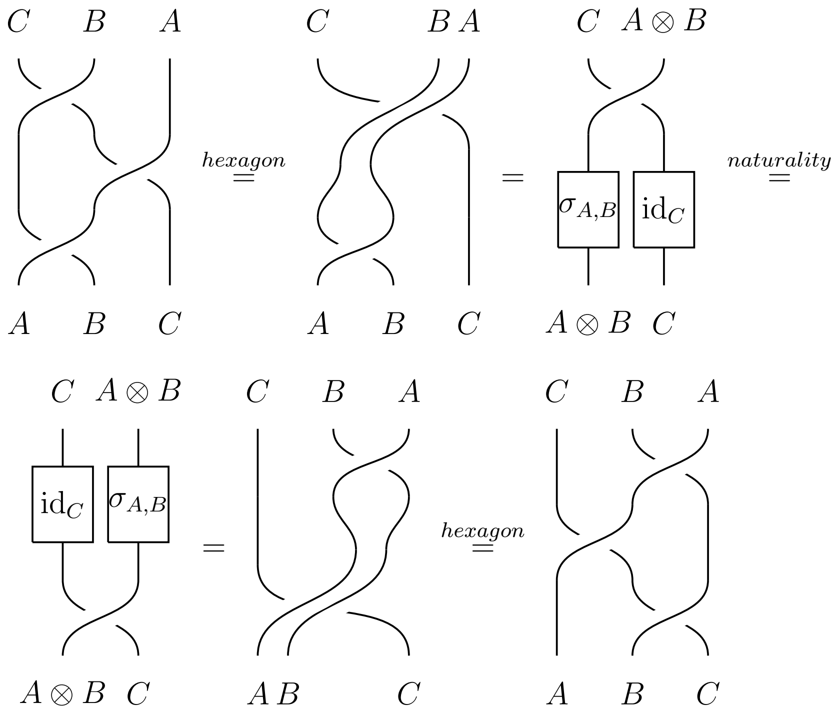
(a) We reason as follows:





Note that when applying naturality, one of the wire types is composite, that is, it is the tensor product of two other types. Thus, we can think of the double wire coming out of the unnamed box as a single wire of type $A \otimes B$. We have also explicitly depicted a box for the identity morphism in order to make the applicaiton of naturality more recognizable.

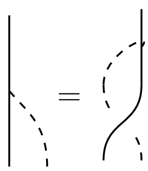
(b) We have the folowing equalities:



(1)

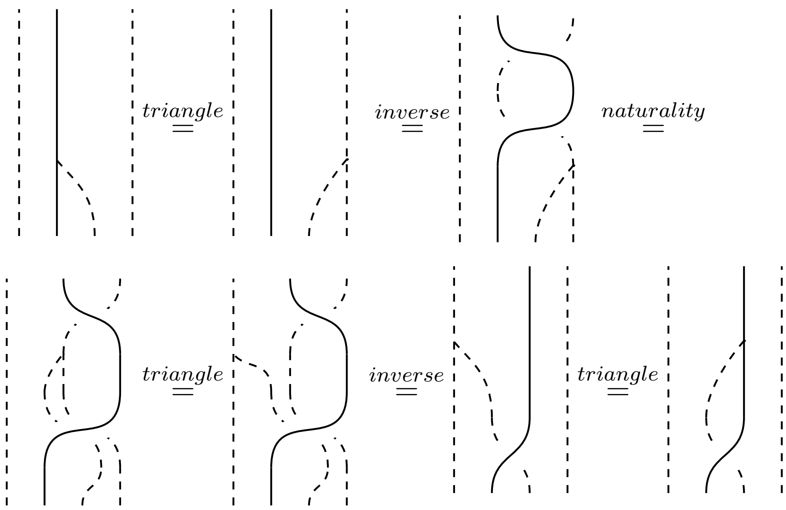
We have written the swap and the identity as boxes to clarify how we use naturality.

(c) We will prove that the following equality holds:



(2)

Composing with λ^{-1} on both sides gives us what we wanted to prove.



(3)

Then the equality follows from invertibility and naturality of ρ and λ.

Exercise 1.4.8

(a) The history $x \circ f \circ a$ occurs with probability

$$|x \circ f \circ a|^2 = |\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}|^2 = \tfrac{1}{2}.$$

The history $y \circ f \circ a$ occurs with probability

$$|y \circ f \circ a|^2 = |\tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}|^2 = \tfrac{1}{2}.$$

Hence it is possible for $|0\rangle$ to evolve to either $|0\rangle$ or $|1\rangle$ under f.

(b) We find that $x \circ f \circ a$ is the nonempty relation $\bullet \mapsto \bullet$ factored as $\bullet \mapsto \{0\} \mapsto \bullet$, and that $y \circ f \circ a$ is the same relation factored as $\bullet \mapsto \{1\} \mapsto \bullet$. This is not the empty relation $\emptyset$, so it is possible for $\{0\}$ to evolve to either $\{0\}$ or $\{1\}$ under f.

Exercise 1.4.9

(a) Choose a terminal object for I. Given objects A and B, choose a product for $A \otimes B$. Given morphisms $f\colon A \to B$ and $f'\colon A' \to B'$, the universal property provides a morphism $f \otimes f\colon A \otimes A' \to B \otimes B'$. The pentagon and triangle equations follow from the fact that the mediating map in the universal property of products is unique.

(b) Given objects A and B, we have to produce a morphism $A \times B \to B \times A$. By the universal property of products, it suffices to produce morphisms $A \otimes B \to B$ and $A \otimes B \to A$. Take the two projection maps.

(c) This follows from the definition of products.