# Introduction to Quantum Programming and Semantics: tutorial 2 answers

**Exercise 2.1.** Let $M$ be a commutative monoid. Then $\otimes\colon M \times M \to M$ defines a functor. The functoriality of $\otimes$ is checked with the commutativity of multiplication as follows.

$$\otimes(m \circ n, m' \circ n') = mnm'n' = mn'm'n = \otimes(m, m') \circ \otimes(n, n')$$

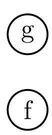
The symmetric monoidal category we construct is strict monoidal; that is, the associator and unitors are identities. (Note that $I = *$.)

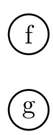
$$\begin{aligned} (* \otimes *) \otimes * &= * = * \otimes (* \otimes *) \\ I \otimes * &= * = * \otimes I \end{aligned}$$

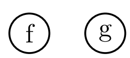
The naturalities of the associator and unitors can be checked easily. For example, the naturality of the associator is checked as follows.

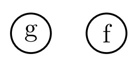
$$\begin{array}{ccc} (* \otimes *) \otimes * & \xrightarrow{\alpha_{*,*,*} = \mathrm{id}} & * \otimes (* \otimes *) \\ {\scriptstyle (m \otimes n) \otimes l = mnl} \downarrow & & \downarrow {\scriptstyle m \otimes (n \otimes l) = mnl} \\ (* \otimes *) \otimes * & \xrightarrow{\alpha_{*,*,*} = \mathrm{id}} & * \otimes (* \otimes *) \end{array}$$

The triangle and pentagon diagrams are also trivial since all the morphisms appearing in the diagrams are identities.

**Exercise 2.2.** (1) 

(2) 

(3) 

(4) 


Note that all of these are equal without the assumption of the monoidal category to be symmetric.

**Exercise 2.3.** If a monoidal category only has one object, then it has to be the tensor unit. Every morphism in the category is a scalar, and from Exercise 2.2, $g \circ f = f \circ g$ for all $f$ and $g$. Therefore, regarding the category as a monoid, it is commutative. From Exercise 2.1, the category is a symmetric monoidal.

**Exercise 2.4.** When working on discrete categories, we do not need to care about commutativity of diagrams since every morphism is identity.
The functor $\otimes\colon M \times M \to M$ sends $(m, n)$ to $mn$. The tensor unit is the unit element of the monoid. Since the oppsite category of a discrete category is itself, the dagger functor is the identity.

**Exercise 2.5.** The only non-trivial one is the positive map. Given a positive map $f$ on a Hilbert space, we first use two Hermitian maps $g$ and $h$ to decompose $f$ as $f = g + ih$. Applying the spectral decomposition to $g$ and $h$, we have

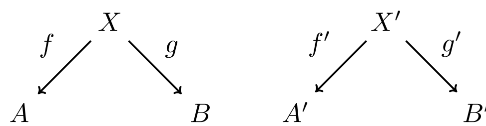
$$g = \sum_i \lambda_i |i\rangle\langle i|, \quad h = \sum_j \mu_j |j\rangle\langle j|.$$

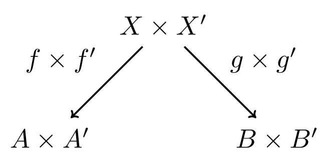
Therefore, we can write $\langle v|f|v\rangle$ as

$$\sum_i \lambda_i |\langle v|i\rangle|^2 + i \sum_j \mu_j |\langle v|j\rangle|^2.$$

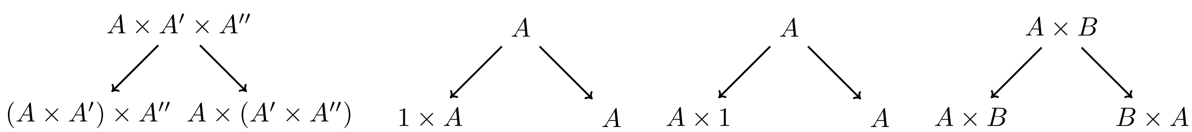
But since this has to be non-negative for all $v$, we have $\lambda_i \geq 0$ and $\mu_j = 0$. Defining $k$ by $\sum_i \sqrt{\lambda_i}|i\rangle\langle i|$, we have $f = k^\dagger \circ k$.

**Exercise 2.6.** The identity of an object $X$ is a span $X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$ in **Set**. One can check that the composition of $n$ morphisms $(X_i, f_i, g_i)\colon A_i \to A_{i+1}$ is defined by the set $\{(x_0, \ldots, x_{n-1}) \mid f_i(x_i) = g_{i+1}(x_{i+1})\}$ with two functions $f(x_0, \ldots, x_n) = f_0(x_0)$ and $g(x_0, \ldots, x_{n-1}) = g_{n-1}(x_{n-1})$.
To make **Span** a monoidal category with tensor product defined by the usual Cartesian product of sets, we would first want to construct the functor $\otimes\colon \mathbf{Span} \times \mathbf{Span} \to \mathbf{Span}$. This functor sends an object $(A, A')$ of $\mathbf{Span} \times \mathbf{Span}$ which is a pair of sets to $A \times A'$, and a morphism $((X, f, g), (X', f', g'))\colon (A, A') \to (B, B')$ to $(X \times X', f \times f', g \times g)$.

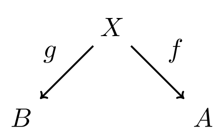


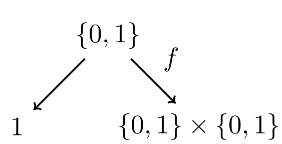

The tensor unit is the singleton 1. Associater, left/right unitor and braiding are defined as canonical spans as follows. [1]




The dagger functor $\mathbf{Span} \to \mathbf{Span}^{\mathrm{op}}$ is defined by flipping the legs $f$ and $g$ of a morphism $(X, f, g)\colon A \to B$.




We also have $(X, f, g)^\dagger \otimes (X', f', g')^\dagger = (X, g, f) \otimes (X', g', f') = (X \times X', f \times f', g \times g') = ((X, f, g) \otimes (X', f', g'))^\dagger$.
An example of an entangled state is the span




where $f$ is defined by $f(0) = (0, 1)$, $f(1) = (1, 0)$.

[1] Strictly speaking, the difference of $(A \times A') \times A''$ and $A \times (A' \times A'')$ has to be ignored somewhat.