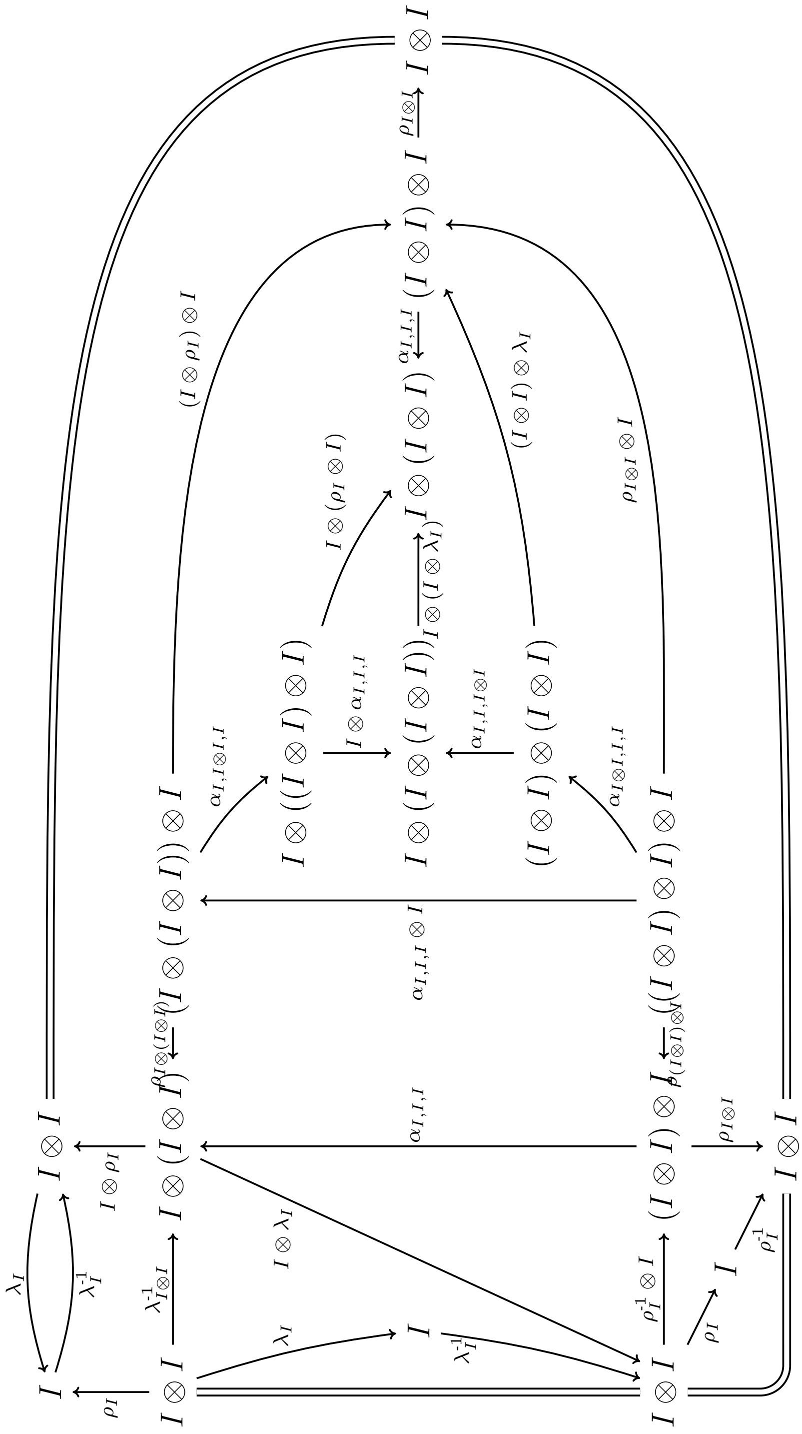
Introduction to Quantum Programming and Semantics: solutions

Exercise 1.13

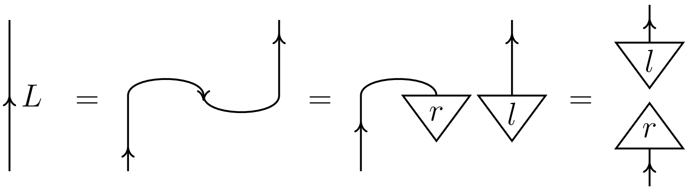
The axiom applying to each region can be deduced from its number of non-identity sides: 2 for invertibility, 3 for triangle, 4 for naturality, and 5 for pentagon.



$I \otimes I$
$(I \otimes I) \otimes I$
$I \otimes (I \otimes I)$
$I \otimes ((I \otimes I) \otimes I)$
$I \otimes (I \otimes (I \otimes I))$
$(I \otimes I) \otimes (I \otimes I)$
$(I \otimes (I \otimes I)) \otimes I$
$((I \otimes I) \otimes I) \otimes I$
I

Exercise 3.1

Suppose that η is the morphism $I \xrightarrow{\lambda_i^{-1}} I \otimes I \xrightarrow{r \otimes l} R \otimes L$. Then



A similar argument holds for id_R.

Interpreting a graphical diagram as a history of events that have taken place, as we do, the fact that id_L factors through I means that, in any observable history of this experiment, whatever input we give the process, the output will be independent of it. Clearly such objects L are quite degenerate.

Exercise 3.4

(a) Like any vector in $R \otimes L$, we can write $\eta(1)$ as $\sum_{j=1}^m z_j x_j \otimes y_j$ for $z_j \in \mathbb{C}$, $x_j \in R$, and $y_j \in L$, where m is some finite number. Developing each x_j on the basis $\{r_i\}$ and using bilinearity of the tensor product, we see that we can also write it as $\sum_{i=1}^n r_i \otimes l_i$ for $n = \dim(V)$ and $l_i \in L$. If we could also write it as $\sum_{i=1}^n r_i \otimes l'_i$, then we would have $0 = \sum_{i=1}^n r_i \otimes (l_i - l'_i)$. Because r_i forms a basis, it would follow that $l_i = l'_i$ for each i. Hence the l_i are unique.

(b) Use the snake equation:

$$\begin{aligned} l &= \mathrm{id}_L(l) \\ &= (\varepsilon \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \eta)(l) \\ &= (\varepsilon \otimes \mathrm{id}_L)(\sum_i l \otimes r_i \otimes l_i) \\ &= \sum_i \varepsilon(l \otimes r_i) l_i. \end{aligned}$$

(c) Similarly, it follows from the snake equation that $r_i = \sum_k \varepsilon(l_k \otimes r_i) r_k$. Suppose that $l_i = l_j$. Because $\{r_k\}$ are linearly independent, then $\varepsilon(l_i \otimes r_i) = 1$, and $\varepsilon(l_k \otimes r_i) = 0$ for $k \neq i$. Hence $\varepsilon(l_j \otimes r_i) = 1$, and it follows that $i = j$, and so $r_i = r_j$. So f is injective.

(d) First notice that the standard form unit and counit indeed satisfy the snake equation. For the converse, combine the previous parts with Lemma 3.5.

Alternatively, pick a basis $\{r_i\}_{i=1}^n$ for R as before. Write $\eta(1)$ as $\sum_{j=1}^m = x_j \otimes y_j$. Now develop $x_j = \sum_{k=1}^n z_{jk} r_k$ on the r_i basis. Then:

$$\eta(1) = \sum_{j=1}^m x_j \otimes y_j$$

$$= \sum_{j=1}^{m} (\sum_{k=1}^{n} z_{jk} r_k) \otimes y_j$$
$$= \sum_{k=1}^{n} r_k \otimes (\sum_{j=1}^{m} z_{jk} y_j)$$

Now take $l_j = z_{jk} y_j$, and apply parts (a), (b), and (c).

Exercise 3.5

(a) A Hilbert space is in particular a vector space. In the previous exercise, we may start by choosing $\{r_i\}$ to be orthonormal.

(b) First, compute that $\eta^\dagger(r_i \otimes l_j) = \langle l_i | l_j \rangle$:

$$\begin{aligned} \langle \eta(1) | r_i \otimes l_j \rangle &= \sum_k \langle r_k | r_i \rangle \langle l_k | l_j \rangle \\ &= \langle l_i | l_j \rangle \\ &= \langle 1 | \eta^\dagger(r_i \otimes l_j) \rangle. \end{aligned}$$

Hence dagger duality shows that $\varepsilon(l_i \otimes r_j) = \eta^\dagger \circ \sigma(l_i \otimes r_j) = \eta^\dagger(r_j \otimes l_i) = \langle l_j | l_i \rangle$. But part (a) shows that also $\varepsilon(l_i \otimes r_j) = \delta_{ij}$. Hence $\langle l_i | l_j \rangle = \delta_{ij}$, making $\{l_i\}$ orthonormal.

Exercise 3.6

First notice that the standard form indeed satisfies the snake equations.

Second, if η and ε witness $L \dashv R$, then for each $r \in R$ there exists $l \in L$ such that $(\bullet, (r, l)) \in \eta$ by one snake equation. But there can be at most one such l because of the other snake equation. More precisely:

$$\begin{aligned} \{(\bullet, l)\} &= (\varepsilon \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \eta)(l, \bullet) \\ &= (\varepsilon \otimes \mathrm{id}_L)(\{l\} \times \eta) \\ &= \bigcup_{(r,l') \in \eta} \varepsilon(l, r) \times \{l'\} \end{aligned}$$

So if $(r, l') \in \eta$ and $(l, r) \in \varepsilon$, then $l = l'$. Thus $f(r) = l$ defines an isomorphism $R \xrightarrow{f} L$ that makes η of the standard form. By Lemma 3.5, also ε must be of the standard form.

Third, observe that if $f \neq f'$, then $\eta \neq \eta'$. Hence different choices of isomorphism $R \simeq L$ yield different (co)unit maps.

Finally, notice that any isomorphism is a unitary.