Introduction to Quantum Programming and Semantics: tutorial 5 answers

Exercise 3.11

Let $X \xrightarrow{R} X$. Compute:

$$\begin{aligned}
\mathrm{Tr}(R) &= \varepsilon \circ (R \otimes \mathrm{id}_X) \circ \sigma_{X,X} \circ \eta \\
&= \{((x,x),\bullet) \mid x \in X\} \circ (R \otimes \mathrm{id}_X) \circ \{((x,y),(y,x)) \mid x,y \in X\} \circ \{(\bullet,(x,x)) \mid x \in X\} \\
&= \{((x,x),\bullet) \mid x \in X\} \circ (R \otimes \mathrm{id}_X) \circ \{(\bullet,(x,x)) \mid x \in X\} \\
&= \{((x,x),\bullet) \mid x \in X\} \circ \{(\bullet,(y,x)) \mid (x,y) \in R\} \\
&= \{(\bullet,\bullet) \mid \exists x \in X \colon xRx\}.
\end{aligned}$$

So $\mathrm{Tr}(R) = 1$ when R has a fixed point, and $\mathrm{Tr}(R) = 0$ otherwise.

Exercise 3.12

(a) Say $f = g^\dagger \circ g$ for $A \xrightarrow{g} B$. Now use dagger duality:

$$\begin{aligned}
\mathrm{Tr}_A(f) &= \varepsilon_A \circ (g^\dagger \otimes \mathrm{id}_{A^*}) \circ (g \otimes \mathrm{id}_{A^*}) \circ \sigma_{A^*,A} \circ \eta_A \\
&= \varepsilon_A \circ (g^\dagger \otimes \mathrm{id}_{A^*}) \circ \sigma_{A^*,B} \circ (\mathrm{id}_{A^*} \otimes g) \circ \eta_A \\
&= \eta_A^\dagger \circ \sigma_{A,A^*} \circ (g^\dagger \otimes \mathrm{id}_{A^*}) \circ \sigma_{A^*,B} \circ (\mathrm{id}_{A^*} \otimes g) \circ \eta_A \\
&= \eta_A^\dagger \circ (\mathrm{id}_{A^*} \otimes g^\dagger) \circ (\mathrm{id}_{A^*} \otimes g) \circ \eta_A.
\end{aligned}$$

(b) If $f = g^\dagger \circ g$, then $f^* = g^* \circ (g^\dagger)^* = (g^*) \circ (g^*)^\dagger$.

(c)

$$\begin{aligned}
\mathrm{Tr}_{A^*}(f^*) &= \varepsilon_{A^*} \circ (f^* \otimes \mathrm{id}_A) \circ \sigma_{A,A^*} \circ \eta_{A^*} \\
&= \varepsilon_{A^*} \circ (\mathrm{id}_{A^*} \otimes f) \circ \sigma_{A,A^*} \circ \eta_{A^*} \\
&= \varepsilon_A \circ \sigma_{A,A^*} \circ (\mathrm{id}_{A^*} \otimes f) \circ \sigma_{A,A^*} \circ \sigma_{A^*,A} \circ \eta_A \\
&= \mathrm{Tr}_A(f).
\end{aligned}$$

(d) This is graphically immediately clear.

(e) Suppose $f = a^\dagger \circ a$ and $g = b^\dagger \circ b$; use the cyclic property to see $\mathrm{Tr}(g \circ f) = \mathrm{Tr}((b^\dagger \circ a)^\dagger \circ (b^\dagger \circ a))$, and then use part (a) to see that this scalar is positive.

Exercise 3.13

If $A \otimes B \xrightarrow{h} A \otimes B$ is an inverse of $f \otimes g$, take f^{-1} to be $\mathrm{Tr}_B((\mathrm{id} \otimes g) \circ h) \bullet \dim(B)^{-1}$. This works in any monoidal category by dropping the braid from the trace/dimension.