Introduction to Quantum Programming and Semantics: solutions

Exercise 5.4

(a) The defining equation for phases gives

$$\{g^{-1} \circ h \mid g, h \in a\} = \{\mathrm{id}_x \mid x \in \mathrm{Ob}(\mathbf{G})\} = \{g \circ h^{-1} \mid g, h \in a\}.$$

Call the left-hand set L, the middle set M and the right-hand set R. The inclusion $L \subseteq M$ means: $\forall g, h \in a\colon \mathrm{cod}(g) = \mathrm{cod}(h) \implies g = h$. The inclusion $M \supseteq R$ means: $\forall g, h \in a\colon \mathrm{dom}(g) = \mathrm{dom}(h) \implies g = h$. In other words: there can be at most one arrow in a out of each object of $\mathbf{G}$, and at most one arrow of a into each object of $\mathbf{G}$. Given this, the remaining inclusions $L \supseteq M \subseteq R$ mean: $\forall x \in \mathrm{Ob}(\mathbf{G}) \exists g, h \in a\colon \mathrm{dom}(g) = x = \mathrm{cod}(h)$. That is: a contains arrows into and out of each object.

(b) Pick an object x; the phase a contains exactly one arrow $x \to y$. If $y = x$, we have a 1-cycle. Otherwise, a contains exactly one arrow $y \to z$, etc. This process has to end, because the groupoid is finite. Delete all the objects involved in the cycle, and repeat.

For the indiscrete groupoid on $\mathbb{Z}$, there is a phase $\{n \xrightarrow{!} n+1 \mid n \in \mathbb{Z}\}$

(c) Combining the fact that G is skeletal with (a), we get that a phase ϕ contains exactly one self-loop for every object in G. That is:

$$\phi = \{f_x \mid x \in \mathrm{Ob}(G), f_x : x \to x\}$$

Next, we define the required group isomorphism in the following way:

$$F(\phi) = \prod_{x \in \mathrm{Ob}(G)} f_x$$

that is we just take the cartesian product (for some ordering of the elements) of the morphisms in ϕ. Note, that we take the standard cartesian product (in $\mathbf{Set}$).

Exercise 5.7

(a) Firstly, note that A is its own dual by virtue of carrying a Frobenius structure. Let us pick the cup/cap pair induced by the Frobenius structure and compute the braided trace by $\mathrm{unit} \circ \mathrm{comonoid} \circ \mathrm{swap} \circ \mathrm{monoid} \circ \mathrm{counit}$; by commutativity, this equals $\mathrm{unit} \circ \mathrm{comonoid} \circ \mathrm{monoid} \circ \mathrm{counit}$, which is the dimension of A. We know that all cup/cap pairs give the same dimension, therefore our choice of these specific ones indeed does not lose generalisation. Now apply the first disconnectedness equality and the diagram becomes $\mathrm{unit} \circ \mathrm{comonoid} \circ (\mathrm{counit} \otimes \mathrm{counit})$. By counitality this equals $\mathrm{unit} \circ \mathrm{counit}$, which by the second disconnectedness equality equals 1.

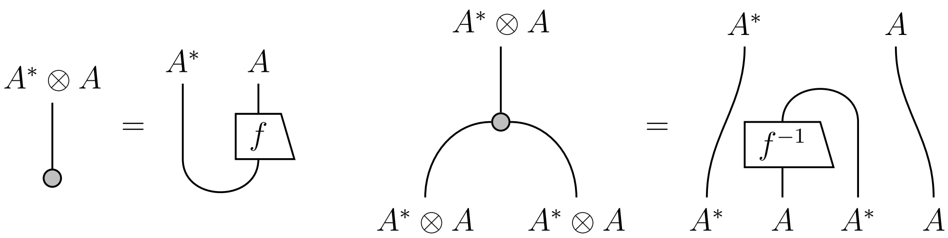
(b) Note from the argument above that the braided dimension equals the dimension, which equals 1. In **Hilb**, this means disconnected Frobenius structures can only live on I.

(c) In **Rel**, it only restricts Frobenius structures to live on nonempty sets.

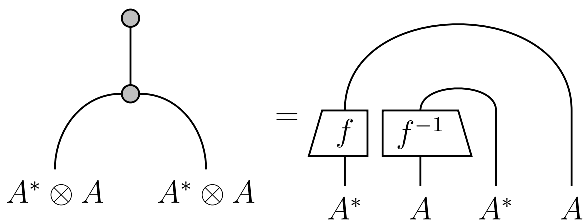
(d) However, for dagger special Frobenius structures, *i.e.* groupoids $\mathbf{G}$, it does follow that G must have precisely one morphism. So disconnected dagger special Frobenius structures in **Rel** can only live on I, too.

Exercise 5.10

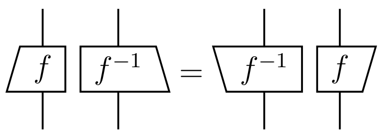
In any dagger compact category, if $f\colon A \to A$ is invertible, then the following defines a dagger Frobenius structure on $A^* \otimes A$:



(To verify both unit laws, use that f^{-1} is both left and right inverse to f.) Now



so this Frobenius structure is symmetric if and only if



But this is not true, for example, for

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad f^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

in **FHilb** because

$$f_* \otimes f^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix} = (f^{-1})^* \otimes f^\dagger$$