**Randomness and Computation 2023**
**Tutorial 5 (week 7)**
**Solutions**

1. **A frog lives in a pond with two lily pads.** The frog toss a coin every morning to decided on which lily pad it will stay the rest of the day. When the coins lands head the frog changes lily pad, when it lands tail the frog remain in the same lily pad. Each lily pad has its own coin, with probability of landing head of $p$ for the left lily pad and probability $q$ for the right lily pad. Let $(X_0, X_1, ....X_t, ...)$ the Markov Chain associated with the sequence of lilly pads occupied by the frog.

   (a) What is its transition probability $P(x, y)$ of this Markov Chain?

   (b) Draw its associated graph.

   (c) Use the detail balance relation to give a candidate of stationary distribution.

   (d) Discuss the case where either $p$ or $q$ is zero. What is the behaviour of the Markov chain and whether the detail balance equation provides a solution.

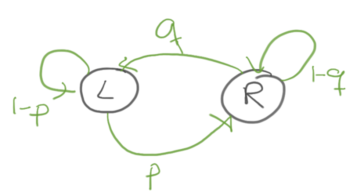
   (e) Discuss the case $p = q = 0$ is zero.

   **Solution:**

   (a) The transition probability reads:

$$P(x, y) = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \tag{1}$$

   (b) This is the associated graph:




   (c) We have $\pi_l p = \pi_r q$, which can be rewritten as $\pi_l/\pi_r = \frac{q}{r}$. Finally, using the normalization condition $\pi_l + \pi_r = 1$, we obtain $\pi_r = \frac{p}{p+q}$ and $\pi_l = \frac{q}{p+q}$.

   (d) When only one of the two $p$ or $q$ is zero, let's say $p = 0$ without loss of generality. When the frog seats one the left lily pad it remains forever. Whenever $p$ or $q$ is zero, the distribution converges to the frog staying in one of the two lily pads with certainty.

   (e) When $p = q = 0$ there is not unique stationary distribution as the frog always remains in the starting position.

2. **Coupon Collector**. A company issues $n$ different types of coupons. A collector desires a complete set. We assume each coupon is equally likely to be acquired. How many coupons must the collector acquire so that the collection contains all $n$ types? . Let $X_t$ be the number of different types represented among the collector's first $t$ acquired coupons ($X_0 = 0$).

   (a) What is its transition probability $P(x, y)$ of this Markov Chain?

   (b) Draw its associated graph.

   (c) Arguing on its structure, what should be its stationary distribution?

   (d) Is the Markov chain irreducible?

   (e) Is it aperiodic?

   (f) Does it satisfies the detail balance condition?

   (g) Let $\tau$ be the random variable corresponding to the number of coupons collected when the collection first contains every type. Proof the following relation:

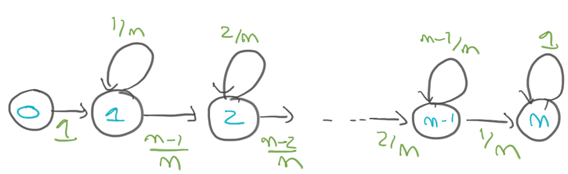
$$\Pr[\tau > \lceil n(\log n + \ln 2 \log \epsilon^{-1})\rceil] \leq \epsilon \tag{2}$$

**Solution:**

   (a) The state satisfy $j \in \{0, 1, ..., n\}$ and the transition probability reads:

$$P(j,k) = \begin{cases} \frac{n-j}{n} & \text{if } k = j+1 \\ \frac{j}{n} & \text{if } k = j \\ 0 & \text{Otherwise.} \end{cases} \tag{3}$$

   (b) This is the associated graph:




   (c) The Markov chain is never decreasing, therefore it will ultimately reach the final state $n$. Therefore, the stationary distribution is $\pi(n) = 1$ and $\pi(i) = 0$ for all $0 \leq j < n$.

   (d) The chain is not irreducible as when in the state $i = n$ it has zero probability to reach any other state. Remark that this not prevent the chain in this case to converge to a stationary distribution $\pi(n) = 1$.

   (e) The chain does not shown any periodicity as the are not cycles other than size 1.

(f) As $P(i+1,i)=0$, as the chain is never decreasing, it is easy to see that detail balance condition is still satisfied as we have $\pi_i P(i,i+1) = \pi(i+1)P(i+1,i) = 0$ as both $\pi_i = 0$ for all $0 \le i < n$.

(g) Let define $A_i$ the event that the $i$-th type does not appears among the first $L = \lceil n(\log n + \ln 2 \log \epsilon)\rceil$ coupons drawn. Using the union bound we obtain

$$\Pr[\tau > L] = \Pr\left[\bigcup_{i=1}^{n} A_i\right] \le \sum_{i=1}^{n} \Pr[A_i]]. \tag{4}$$

Since each trial has probability $1 - 1/n$ of not drawing coupon $i$ and the trials are independent, we get

$$\Pr[\tau > L] \le \sum_{i=1}^{n}\left(1-\frac{1}{n}\right)^{L} \le n e^{-\frac{n\log n + n \ln 2 \log \epsilon^{-1}}{n}} = e^{\ln 2 \log 1/\epsilon} = \epsilon \tag{5}$$

3. **Random walk on a graph** $G$. Consider a simple random walk on a strongly connected undirected graph $G = (V, E)$ where the probability of transitioning from vertex $x$ to $y$ connected by edge $e = (x, y)$ is inversely proportional to the degree of vertex $x$.

   (a) Use the detail balance condition to obtain a candidate stationary distribution on $V$.

   (b) If the graph $G$ happens to be associated to a periodic Markov chain, what trick we can use to transform the walk into one that is associated with an aperiodic Markov chain. What is the stationary distribution that this new Markov chain converges to?

**Solution:**

(a) Using the notation $d(v)$ for the degree of vertex $v \in V$. Detail balance leads to

$$\pi(x)\frac{1}{d(x)} = \pi(y)\frac{1}{d(y)}, \tag{6}$$

which gives the relation

$$\sum_{x\in V}\pi(x) = 1 = \sum_{x\in V} d(x)\pi(y)\frac{1}{d(y)}. \tag{7}$$

Using $\sum_{x\in V} d(x) = 2|E|$ we obtain finally

$$\pi(y) = \frac{d(y)}{2|E|}. \tag{8}$$

(b) We can always transform the Markov chain on graph $G$ into its lazy version, which add a positive probability of remaining in the same vertex. Interestingly, this does not change the detail balance relation and the derivation above, which show that $\pi$ derived above will then be the stationary distribution of the lazy Markov chain.

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