Randomness and Computation 2023
Tutorial 5 (week 7)
Issue date: Wednesday, 25th October, 2023.

1. **A frog lives in a pond with two lily pads.** The frog tosses a coin every morning to decide on which lily pad it will stay for the rest of the day. If the coin lands head, the frog switches to the other lily pad, or otherwise, it lands tail and the frog remains on the same lily pad. Each lily pad has its own coin, with probability of landing head of p for the left lily pad and probability q for the right lily pad. Let $(X_0, X_1, ...X_t, ...)$ the Markov Chain associated with the sequence of lilly pads occupied by the frog.

   (a) Give the transition matrix $P(x, y)$ of this Markov Chain.
   (b) Draw its associated graph.
   (c) Use the detailed balance condition to identify a candidate of stationary distribution.
   (d) Consider the case where either p or q is zero. What is the behaviour of the Markov chain, and does the detailed balance equation provide a solution or not?
   (e) Discuss the item above for the case where both p and q are zeros.

2. **Coupon Collector**. A company issues n different types of coupons. A collector desires a complete set. We assume each coupon is equally likely to be acquired. How many coupons must the collector acquire so that the collection contains all n types? . Let X_t be the number of different types represented among the collector's first t acquired coupons ($X_0 = 0$).

   (a) What is its transition probability $P(x, y)$ of this Markov Chain?
   (b) Draw its associated graph.
   (c) Arguing on its structure, what should be its stationary distribution?
   (d) Is the Markov chain irreducible?
   (e) Is it aperiodic?
   (f) Does it satisfies the detail balance condition?
   (g) Let τ be the random variable corresponding to the number of coupons collected when the collection first contains every type. Proof the following relation:

   $$\Pr[\tau > n(\log n + \ln 2 \log \epsilon)] \leq \epsilon \tag{1}$$

3. **Random walk on a graph** G. Consider a simple random walk on a connected undirected graph $G = (V, E)$ where the probability of transition from vertex x to y connected by edge $e = (x, y)$ is inversely proportional to the degree of vertex x.

   (a) Use the detailed balance condition to obtain a candidate stationary distribution on V.
   (b) If the graph G happens to be associated to a periodic Markov chain, what trick we can use to transform the walk into one that is associated with an aperiodic Markov chain? What is the stationary distribution that this new Markov chain converges to?

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