Randomness and Computation 2023
Tutorial 7 (week 9)
Solution

1. **Mixing time of a frog living in a pond with two lily pads.** Let's return to the scenario of problem 1 of tutorial 5 of a random walk of a frog between two lily pads.

   The frog tosses a coin every morning to decide on which lily pad it will stay for the rest of the day. If the coin lands head, the frog switches to the other lily pad, or otherwise, it lands tail and the frog remains on the same lily pad. Each lily pad has its own coin, with probability of landing head of p for the left lily pad and probability q for the right lily pad. Let $(X_0, X_1, ...X_t, ...)$ the Markov Chain associated with the sequence of lilly pads occupied by the frog. The transition matrix of the Markov chain reads:

   $$P(x, y) = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

   and its stationary distribution is $\pi(r) = \frac{p}{p+q}$ and $\pi(l) = \frac{q}{p+q}$.

   (a) Given the distribution $P^t(x, y)$ resulting from applying t iteration of the Markov chain to the input x (resulting in output y), prove that its total variation distance with the stationary distribution π satisfies the relation:

   $$\Delta_x^t = \|P^t(x, \cdot) - \pi\| = |P^t(x, r) - \pi(r)| = |P^t(x, l) - \pi(l)|$$

   (b) Write Δ_x^{t+1} as a function of Δ_x^t.

   (c) Write Δ_x^{t+1} as a function of Δ_x^0.

   (d) What is the mixing time of this this Markov chain?

   (e) Discuss the pathological cases where there is no mixing.

   (f) Non-examinable additional question: Compute the eigenvalues of the matrix P and compare to the contraction value of Δ_x^t.

   Solution:

   (a) We have

   $$\Delta_x^t = \|P^t(x, \cdot) - \pi\| = \frac{1}{2}\left(|P^t(x, r) - \pi(r)| + |P^t(x, l) - \pi(l)|\right).$$

   The relations $\pi(r) + \pi(l) = 1$ and $P^t(x, r) + P^t(x, l) = 1$ (there is only two possibles states), allows us to write

   $$|P^t(x, r) - \pi(r)| = |P^t(x, l) - \pi(l)|,$$

   which combined with the initial equation leads to the solution.

(b) It is easy to see that $P^{t+1}(x,r) = (1-p)P^t(x,r) + qP^t(x,l) = (1-p)P^t(x,r) + q(1-P^t(x,r))$. Therefore, one can write

$$\Delta_x^{t+1} = |P^{t+1}(x,\cdot) - \frac{q}{p+q}| = |(1-(p+q))P^t(x,r) - \frac{p}{p+q} + q| = |(1-(p+q))(P^t(x,r) - \pi(r))| = \Delta_x^t.$$

(c) One can easily see that $\Delta_x^{t+1} = (1-(p+q))\Delta_x^t$ implies, by iteration, $\Delta_x^{t+1} = (1-(p+q))^t\Delta_x^0$.

(d) We will have mixing when $Delta_x^t \leq \epsilon$ for both $x = \{r, l\}$. Because $\Delta_x^0 \leq 1$ by definition of total variation distance, we have the general bound $\Delta_{r,l}^t \leq (1-(p+q))^t \leq e^{-(p+q)t}$. This will be $\epsilon-$small when

$$T \geq \frac{1}{p+q} \log \frac{1}{\epsilon},$$

Therefore $\tau(\epsilon) = 1/(p+q)\log(1/\epsilon)$.

(e) When $p = q = 0$ the total variation distance does not contract as there is no change from the initial state of the chain. When $p = q = 1$ there is always change of state, and the total variation distance also does not contract.

(f) The eigenvalues λ of P should satisfy the relation:

$$\det \begin{pmatrix} 1-p-\lambda & p \\ q & 1-q-\lambda \end{pmatrix} = 0$$

After some standard linear algebra we find two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1-(p+q)$. The first is related to the fact that $\pi P = \pi$, i.e., the stationary distribution is an eigenvector of P of eigenvalue $\lambda_1 = 1$. The second eiganvalue $\lambda_2 = 1-(p+q)$ satisfying $|\lambda_2| < 1$ is a condition for mixing for all irreducible and aperiodic Markov chains and also can be related to the mixing time $\tau(\epsilon) = \frac{1}{\lambda_2}\log(1/(\epsilon\pi_{\min}))$, where $\pi_{\min} = \min_x \pi(x)$. This non-examinable material is developed further in Chapter 12 and 13 of Levin and Peres book from the list of references in the course material.

2. **Glauber dynamics for Gibbs distribution on independent sets.** Let $G = (V, E)$ be a graph with maximum degree Δ, Ω be the set of independent sets on G, and $x \in \{0,1\}^V$ a binary encoding of the vertices composing a given independent set. Let $\pi(x)$ the Gibbs distribution on independent set, reads defined by

$$\pi(x) = \begin{cases} \frac{\lambda^{|x|}}{Z(\lambda)} & \text{if } x(v)x(w) = 0 \quad \forall\{v,w\} \in E \\ 0 & \text{Otherwise.} \end{cases} \tag{1}$$

where $|x|$ is the Hamming weight of configuration x, i.e., $|x| = \sum_{v \in V} x(v)$ and $Z(\lambda) = \sum_{x \in X} \lambda^{|x|}$ normalizes π.

The Glauber dynamics updates configuration X_t to a new configuration X_{t+1} by first selecting a vertex $v \in V$ uniformly at random and then implementing an update.

(a) Using the general definition of a Glauber dynamic update, show that we obtain the following update. First, set $X_{t+1}(w) = X_t(w) \quad \forall w \neq v$. Then, if exist $w' \in N(v)$, where $N(v)$ is the neighborhood of v, such that $X_t(w') = 1$.

- we set (M1) [1] $X_{t+1}(v) = 0$
- otherwise (M2)

$$X_{t+1}(v) = \begin{cases} 1 & \text{with probability } \lambda/(1+\lambda), \\ 0 & \text{with probability } 1/(1+\lambda). \end{cases} \tag{2}$$

[20 marks]

Solution The Glauber update starts selecting a vertex $v \in V$. For $x \in \Omega$ and $v \in V$ we define the set of independent set agreeing with x everywhere except possibly at v

$$\Omega(x, v) = \{y \in \Omega : y(w) = x(w) \quad \forall w \neq v\},$$

and $\pi(\Omega(x, y)) = \sum_{y \in \Omega(x,v)} \pi(y)$. Then the transition probability reads:

$$p(x, y) = \pi(y|\Omega(x, v)) \begin{cases} \frac{\pi(y)}{\pi(\Omega(x,v))} & \text{if } y \in \Omega(x, v), \\ 0 & \text{if } y \notin \Omega(x, v). \end{cases} \tag{3}$$

Because the update can only happen in v, we have trivially $X_{t+1}(w) = X_t(v) \quad \forall v \neq w$. Because the update only affect vertex v if there is at least a neighbor of v, i.e., $w' \in N(v)$ in state x that is 1, then $y(v)$ can only be 0, which implies the move M1. In the case where there are no neighbors to v belonging to the independent set $y(v)$ can take two valid values 0 and 1. Let's call it associated configurations y_0 and y_1 respectively, we then have:

$$\Omega(x, v) = \frac{\lambda^{|x|}}{Z(\lambda)} + \frac{\lambda^{|x|+1}}{Z(\lambda)},$$

where the $|x| + 1$ result from the update having one more element in the independent set. Finally this leads to

$$p(x, y_0) = \frac{\lambda^{|x|}/Z(\lambda)}{\lambda^{|x|}/Z(\lambda) + \lambda\lambda^{|x|}/Z(\lambda)} = \frac{1}{1+\lambda}$$

and

$$p(x, y_1) = \frac{\lambda\lambda^{|x|}/Z(\lambda)}{\lambda^{|x|}/Z(\lambda) + \lambda\lambda^{|x|}/Z(\lambda)} = \frac{\lambda}{1+\lambda},$$

which is the update rule of move M2.

(b) Show that the Markov chain associated with this Glauber dynamics is irreducible.

Solution:

(c) Show that the Markov chain associated with this Glauber dynamics is aperiodic.

Solution:

[1] Remark that the labeling (M1) and (M2) is used to facilitate the discussion of the solution.

(d) Show that the Markov chain given this the Glauber dynamics and the distribution π above satisfy the detail balance condition ($\pi(x)P(x,y) = \pi(y)P(y,x)$).

Solution: Because each vertex has only two possible configurations, i.e., 0 or 1, and for a transition to have non-zero probability, x and y can only differ in one vertex, i.e., we have $|x| = |y| \pm 1$. Let's assume, without lost of generality that $|x| = |y| - 1$, i.e., y is larger by one unit. We can then shown that if $x \neq y$, the transition from x to y involves selecting vertex v uniformly at random followed by adding vertex v to the IS x, leading to

$$\pi(x)P(x,y) = \pi(x)\frac{1}{|V|}p(x,y) = \frac{1}{|V|}\frac{\lambda^{|x|}}{Z(\lambda)}\frac{\lambda}{1+\lambda} = \frac{1}{|V|}\frac{\lambda^{|x|+1}}{Z(\lambda)}\frac{1}{1+\lambda} = \pi(y)\frac{1}{|V|}p(y,x) = \pi(y)P(y,x),$$

which is the same as selecting vertex v uniformly at random followed removing vertex v from the IS y that leads to x.

3. **Metropolis and Glauber for q-coloring.** Fix a set of colors $\{1, 2, ..., q\}$. A proper $q-$coloring of a graph $G = (V, E)$ is an assignment of colors to vertices V, subject to the constraint that neighboring vertices do not receive the same color.

(a) Build a Metropolis algorithm that only allows transitions between coloring differing at a single vertex and that has as stationary distribution the uniform distribution over the set of q-colorings of G.

Solution An obvious choice of Metropolis algorithm consist on first choosing uniformly at random a vertex v and secondly chose also uniformly at random a color c among the all q colors. The Metropolis rule implies that X_t is updated to the new coloring if the proposed color generates a proper coloring of G, otherwise we reject.

(b) Build a Glauber dynamics that only allows transition between coloring differing at a single vertex and that has as stationary distribution the uniform distribution over the set of q-colorings of G.

Solution A Glauber dynamics will also choose a uniformly at random a vertex v. Now we will choose at random one color among the allowed colors at vertex v, which can be easily obtain by inspection of the adjacent vertices. Note that transition are permitted only among coloring's differing at a single vertex. If states x and y agree everywhere except in v, the the chance of moving from x to y equals $1/(|V||A_v(y)|)$, where $A_v(x)$ is the set of allowed colors at v in state x. Since $A_v(x) = A_v(y)$, as they share same neighbors, the probabilities of moving from x to y match the one from y to x and using detailed balance we can prove that the stationary distribution is the uniform distribution. Remaek that irreversibility and aperiodicity are easily proven following the same argument as in Problem 1.

(c) Discuss the differences between the Metropolis and Glauber dynamics for this specific $q-$coloring problem.

Solution We observe that the Metropolis algorithm and Glauber dynamics are slightly different, for example the probability of remaining in the same coloring differs for the two

MC. If there are a allowable color at vertex v selected for updating, Glauber dynamics gives a probability of $1/a$ of remaining, where Metropolis gives $(1+q-a)/q$, which is equal only if $q=a$.

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