Randomness and Computation 2023
Tutorial 7 (week 9)
Issue date: Friday, 10th November, 2023.

1. **Glauber dynamics for Gibbs distribution on independent sets.** Let $G = (V, E)$ be a graph with maximum degree Δ, Ω be the set of independent sets on G, and $x \in \{0,1\}^V$ a binary encoding of the vertices composing a given independent set. Let $\pi(x)$ the Gibbs distribution on independent sets:

$$\pi(x) = \begin{cases} \frac{\lambda^{|x|}}{Z(\lambda)} & \text{if } x(v)x(w) = 0 \quad \forall\{v,w\} \in E \\ 0 & \text{Otherwise.} \end{cases} \tag{1}$$

where $|x|$ is the Hamming weight of configuration x, i.e., $|x| = \sum_{v \in V} x(v)$ and $Z(\lambda) = \sum_{x \in X} \lambda^{|x|}$ normalizes π.

The Glauber dynamics updates configuration X_t to a new configuration X_{t+1} by first selecting a vertex $v \in V$ uniformly at random and then implementing an update.

(a) Using the general definition of a Glauber dynamic update, show that we obtain the following update rule. First, set $X_{t+1}(w) = X_t(w) \quad \forall w \neq v$.
Then, if exist $w' \in N(v)$, where $N(v)$ is the neighborhood of v, such that $X_t(w') = 1$.
- we set (M1) [1] $X_{t+1}(v) = 0$
- otherwise (M2)

$$X_{t+1}(v) = \begin{cases} 1 & \text{with probability } \lambda/(1+\lambda), \\ 0 & \text{with probability } 1/(1+\lambda). \end{cases} \tag{2}$$

(b) Show that the Markov chain associated with this Glauber dynamics is irreducible.
(c) Show that the Markov chain associated with this Glauber dynamics is aperiodic.
(d) Show that the Markov chain associated with this Glauber dynamics and the distribution π above satisfy the detailed balance condition ($\pi(x)P(x,y) = \pi(y)P(y,x)$).

2. **Mixing time of a frog living in a pond with two lily pads.** Let's return to the scenario of problem 1 of tutorial 5 of a random walk of a frog between two lily pads.

The frog tosses a coin every morning to decide on which lily pad it will stay for the rest of the day. If the coin lands head, the frog switches to the other lily pad, or otherwise, it lands tail and the frog remains on the same lily pad. Each lily pad has its own coin, with probability of landing head of p for the left lily pad and probability q for the right lily pad. Let $(X_0, X_1, ...X_t, ...)$ the Markov Chain associated with the sequence of lilly pads occupied by the frog. The transition matrix of the Markov chain reads:

$$P(x,y) = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

and its stationary distribution is $\pi(r) = \frac{p}{p+q}$ and $\pi(l) = \frac{q}{p+q}$.

[1] Remark that the labeling (M1) and (M2) is used to facilitate the discussion of the solution.

(a) Given the distribution $P^t(x, y)$ resulting from applying t iteration of the Markov chain to the input x (resulting in output y), prove that its total variation distance with the stationary distribution π satisfies the relation:

$$\Delta_x^t = \|P^t(x, \cdot) - \pi\| = |P^t(x, r) - \pi(r)| = |P^t(x, l) - \pi(l)|$$

(b) Write Δ_x^{t+1} as a function of Δ_x^t.

(c) Write Δ_x^{t+1} as a function of Δ_x^0.

(d) What is the mixing time of this this Markov chain?

(e) Discuss the pathological cases where there is no mixing.

(f) Non-examinable additional question: Compute the eigenvalues of the matrix P and compare to the contraction value of Δ_x^t.

3. **Metropolis and Glauber for q-coloring.** Fix a set of colors $\{1, 2, ..., q\}$. A proper $q-$coloring of a graph $G = (V, E)$ is an assignment of colors to vertices V, subject to the constraint that neighboring vertices do not receive the same color.

   (a) Build a Metropolis algorithm that only allows transition between coloring differing at a single vertex and that has as stationary distribution the uniform distribution over the set of q-colorings of G.

   (b) Build a Glauber dynamics that only allows transition between coloring differing at a single vertex and that has as stationary distribution the uniform distribution over the set of q-colorings of G.

   (c) Discuss the differences between the Metropolis and Glauber dynamics for this specific $q-$coloring problem.

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