Randomized Algorithms
Tutorial Sheet 2 (Week 4)

1. Let X be a random variable denoting the number of fixed points of a random permutation of $[n] = \{1, \ldots, n\}$, chosen uniformly at random from all $n!$ such permutations. In the lectures (lecture 4) we have calculated that $\mathrm{E}[X] = 1$. What is $\mathrm{Var}[X]$?

2. Suppose our goal is to sample the value of a fair coin ("heads" having probability $1/2$, and "tails" having probability $1/2$) but unfortunately we only have a faulty coin, which returns "heads" with some *unknown* probability p, and returns "tails" with probability $1 - p$.

   Suppose we do not know the value of p, nor even whether $p > 1/2$ or $p < 1/2$. We do assume $p \in (0, 1)$ (neither 0 nor 1). Design a simple algorithm which uses *several* coin flips to return a value ("heads" or "tails") in such a way that "heads" and "tails" are each returned with probability exactly $1/2$.

   (a) Describe the algorithm.

   (b) Argue/prove that "heads" and "tails" are equally likely to be returned.

   (c) Calculate the *expected* number of coin-flips that will be used by your algorithm. You should aim to minimize this. For example, a good algorithm would need $\frac{1}{p(1-p)}$ flips in expectation.

3. Consider the following game. We start with one black ball and one white ball in a bag. We then repeatedly do the following: choose one of the balls in the bag uniformly at random, look at its colour, then put that ball back in the bag together with another ball of the same colour. Repeat this until there are n balls in the bag, for some positive integer $n \geq 2$. Show that the number of white balls is equally likely to be any number between 1 and $n - 1$. In other words, that each possibility has probability $\frac{1}{n-1}$.
   Hint: consider a proof by induction on n.

4. Let Y be a nonnegative integer-valued random variable with (strictly) positive expectation. Prove that

$$\frac{(\mathrm{E}[Y])^2}{\mathrm{E}[Y^2]} \;\leq\; \Pr[Y \neq 0] \;\leq \mathrm{E}[Y].$$

Kousha Etessami