Randomized Algorithms
Tutorial Sheet 4

1. (This is Question 6.1 in the course textbook [MU].)

   Consider the following problem: we are given as input a k-CNF boolean formula with m clauses, over n boolean variables $x_1, \ldots, x_n$, and where every clause has exactly k literals.

   (a) Give a Las Vegas algorithm that finds a truth assignment to the variables that satisfies at least $m(1 - 2^{-k})$ clauses. Analyze the expected running time of the algorithm (and in particular show that it runs in expected polynomial time).

   (b) Give a derandomized algorithm, using the method of conditional expectations.

2. (This is Question 6.3 in the course textbook [MU].)

   Given as n-vertex undirected graph $G = (V, E)$, with $V = \{1, \ldots, n\}$, consider the following randomized method for generating an independent set of vertices in G. (Recall that an *independent set* $I \subseteq V$ is a set of vertices no two of which have an edge between them.).

   For each vertex $i \in V$, let $d_i \in \mathbb{N}$ denote the *degree* of vertex i (i.e., the number of edges incident on vertex i). For any permutation σ of the vertices V, (in other words, for any sequentially ordered listing σ of the n numbers $\{1, \ldots, n\}$), let $S(\sigma) \subseteq V$ be the set of vertices defined as follows:

   $$S(\sigma) = \{i \in V \mid \text{ every neighbor of } i \text{ in } G \text{ occurs after } i \text{ in the permutation } \sigma\}$$

   (a) Show that $S(\sigma)$ is an independent set of $G = (V, E)$, for any permutation σ of V.

   (b) Describe a randomized algorithm for generating a u.a.r. random permutation σ of V. Show that the expected size of $S(\sigma)$ for such a random permutation σ is given by:

   $$\sum_{i=1}^{n} \frac{1}{d_i + 1}$$

   (c) Use this to prove that G must have an independent set of size at least $\sum_{i=1}^{n} \frac{1}{d_i+1}$.

3. In lectures, we only covered the "symmetric" case of the Lovasz Local Lemma, where the probability of all bad events E_i is upper bounded by the *same* probability p. We did this because for many applications of the Local Lemma the symmetric case suffices.

   Here is a general, asymmetric, form of the Lovasz Local Lemma (described in Section 6.9 of the textbook), which allows different upper bounds on the probability of each bad event E_i:

   Theorem. *(Thm 6.17 in the textbook) Let $E_1, \ldots, E_n$ be a set of events, and let $G = (V, E)$ be a dependency graph for these events. Suppose there exists $x_1, \ldots, x_n \in (0, 1)$ such that for all $1 \leq i \leq n$,*

   $$\Pr[E_i] \leq x_i \prod_{(i,j) \in E} (1 - x_j)$$

Then

$$\Pr(\bigcap_{i=1}^{n} \overline{E}_i) \;\geq\; \prod_{i=1}^{n}(1 - x_i) \;>\; 0 \;. \qquad\qquad \Box$$

Consider a variant of the more basic (symmetric) Lovasz Local Lemma, which we covered in lectures, with the following slight modification:

- replace the condition " $4dp \leq 1$ " with the condition " $e \cdot p(d+1) \leq 1$ ".

Here $e = 2.71828\ldots$ denotes the base of the natural logarithm.

(a) Firstly, observe that in cases of a dependency graph with maximum out-degree $d \geq 3$, the condition $e \cdot p(d+1) \leq 1$ is actually the less stringent assumption, meaning it is implied by the assumption $4dp \leq 1$.

(b) Secondly, show that the modified version of the symmetric Lovasz Local Lemma, with the condition $e \cdot p(d+1) \leq 1$, can be derived as a special case of the general (asymmetric) Lovasz Local Lemma (Thm 6.17) re-stated above.
(*Hint.* You may use the following general **Fact**: for any $y \geq 1$,

$$\left(1 - \frac{1}{y+1}\right)^{y} \geq \frac{1}{e} \;. \qquad\qquad)$$

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