Reinforcement Learning

Dynamic Programming (part 2) and Monte Carlo Methods

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Based on slides by Stefano V. Albrecht

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informatics

Lecture Outline

- Value Iteration
- Dynamic programming (part 2)
- DP examples

- Monte Carlo policy evaluation
- Monte Carlo control with...
    - Exploring starts
    - Soft policies
    - Off-policy learning
- Importance sampling

Policy Iteration and Value Iteration

Policy Iteration

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
   $\quad \Delta \leftarrow 0$
   $\quad$ For each $s \in \mathcal{S}$:
   $\quad\quad v \leftarrow V(s)$
   $\quad\quad V(s) \leftarrow \sum_{s',r} p(s',r|s,\pi(s))\big[r + \gamma V(s')\big]$
   $\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $\quad a \leftarrow \pi(s)$
   $\quad \pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
   $\quad$ If $a \neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return V and π; else go to 2

Value Iteration

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

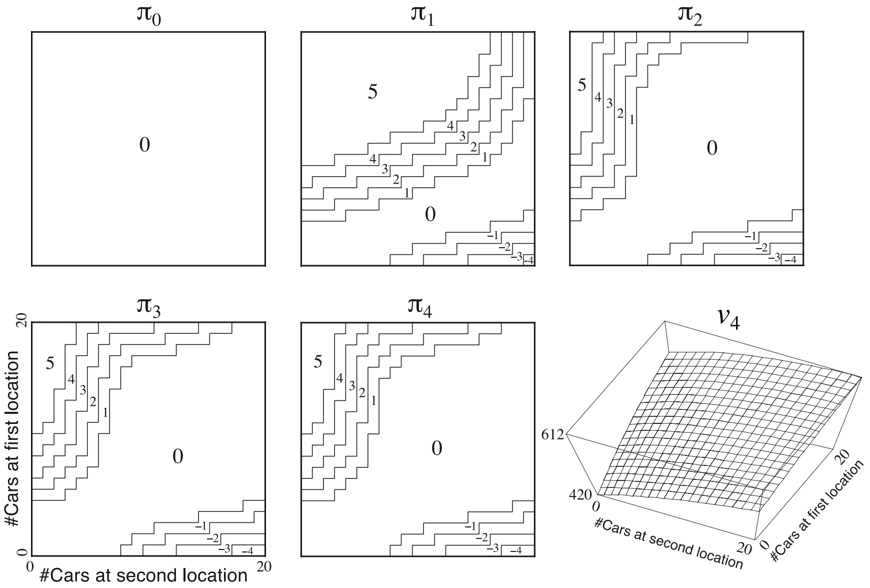
Repeat
$\quad \Delta \leftarrow 0$
$\quad$ For each $s \in \mathcal{S}$:
$\quad\quad v \leftarrow V(s)$
$\quad\quad V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
$\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, π, such that
$\quad \pi(s) = \arg\max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$

Example: Jack's Car Rental

- Two car rental locations
- Cars are requested and returned randomly based on a distribution (see book)
- States: (n_1, n_2) where n_i is number of cars at location i (max 20 each)
- Actions: number of cars moved from one location to other (max 5)
  (positive is from location 1 to 2, negative is from 2 to 1)
- Rewards:
  +\$10 per rented car in time step
  −\$2 per moved car in time step
- $\gamma = 0.9$

Example: Jack's Car Rental



Value Iteration

Iterative policy evaluation may take many sweeps $v_k \rightarrow v_{k+1}$ to converge

Do we have to wait until convergence before policy improvement?

$k = 3$

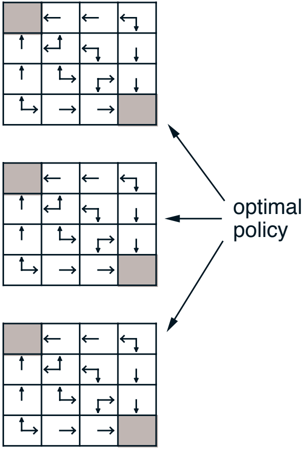
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Value Iteration

Iterative policy evaluation uses Bellman equation as operator:

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \left[r + \gamma v_k(s')\right] \quad \text{for all } s \in \mathcal{S}$$

Value iteration uses *Bellman optimality equation* as operator:

$$v_{k+1}(s) = \max_a \sum_{s',r} p(s', r|s, a) \left[r + \gamma v_k(s')\right] \quad \text{for all } s \in \mathcal{S}$$

- Combines one sweep of iterative policy evaluation and policy improvement
- Sequence converges to optimal policy
  (can show that Bellman optimality operator is γ-contraction)

Value Iteration

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
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until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, π, such that
    $\pi(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$

Asynchronous Dynamic Programming

DP methods so far perform exhaustive *sweeps*:

Policy evaluation and improvement for all $s \in \mathcal{S} \;\Rightarrow\;$ prohibitive if state space large!

Asynchronous DP methods evaluate and improve policy on subset of states:

- Gives flexibility to choose best states to update

  $\Rightarrow$ e.g. random states, recently visited states (real-time DP)

- Can perform updates *in parallel* on multiple processors

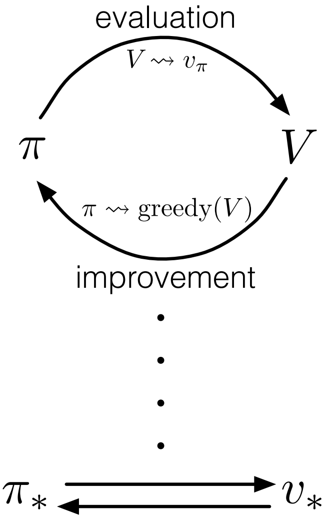
- Still guaranteed to converge to optimal policy if all states in $\mathcal{S}$ are updated infinitely many times in the limit

Conclusion on Dynamic Programming

DP methods iterate through policy evaluation and improvement until convergence to optimal value function v_* and policy π_*

- Policy evaluation via repeated application of Bellman operator
- Requires complete knowledge of MDP model: $p(s', r|s, a)$

Can we compute optimal policy without knowledge of complete model?



Monte Carlo Policy Evaluation

Monte Carlo (MC) methods learn value function based on experience

- Experience: entire episodes $E^i = < S_0^i, A_0^i, R_1^i, S_1^i, A_1^i, R_2^i, ..., S_{T_i}^i >$

MC does not require complete model $p(s', r|s, a)$, only requires sampled episodes

Two ways to obtain episodes:

- **Real experience:** generate episodes directly from “real world”
- **Simulated experience:** use simulation model $\hat{p}$ to sample episodes
  — $\hat{p}(s, a)$ returns a pair (s', r) with probability $p(s', r|s, a)$

Monte Carlo Policy Evaluation

Monte Carlo (MC) Policy Evaluation:

- Estimate value function by averaging sample returns:

$$v_\pi(s) \doteq \mathbb{E}_\pi\left[\sum_{k=t}^{T-1} \gamma^{k-t} R_{k+1} | S_t = s\right] \approx \frac{1}{|\mathcal{E}(s)|} \sum_{t_i \in \mathcal{E}(s)} \sum_{k=t_i}^{T_i - 1} \gamma^{k-t_i} R^i_{k+1}$$

where for each past episode $E^i = < S^i_0, A^i_0, R^i_1, S^i_1, A^i_1, R^i_2, ..., S^i_{T_i} >$:

— **First-visit MC:** $\mathcal{E}(s)$ contains *first* time t_i for which $S^i_{t_i} = s$ in E^i

— **Every-visit MC:** $\mathcal{E}(s)$ contains *all* times t_i for which $S^i_{t_i} = s$ in E^i

- Both methods converge to $v_\pi(s)$ as $|\mathcal{E}(s)| \to \infty$

First-Visit Monte Carlo Policy Evaluation

See Tutorial 5

Initialize:

- $\pi \leftarrow$ policy to be evaluated
- $V \leftarrow$ an arbitrary state-value function
- $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Repeat forever:

- Generate an episode using π
- For each state s appearing in the episode:
  - $G \leftarrow$ return following the first occurrence of s
  - Append G to $Returns(s)$
  - $V(s) \leftarrow \text{average}(Returns(s))$

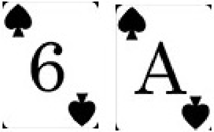
Example: Blackjack

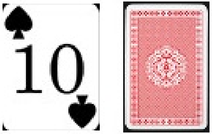
Initial state:

Player

Dealer

Ace worth 1 or 11





Hidden card

First, player samples cards from deck (hit) until stop (stick)

Then, dealer samples cards from deck (hit) until sum > 16 (stick)

Player loses (-1 reward) if bust (card sum > 21)
Player wins (+1 reward) if Dealer bust or Player sum > Dealer sum

Example: Blackjack

Player policy π:

stick if player sum is 20
or 21, else hit

Estimate of v_π using MC ...

States s (3-tuple):

– Player sum (12–21)
– Dealer card (ace–10)
– Usable ace?

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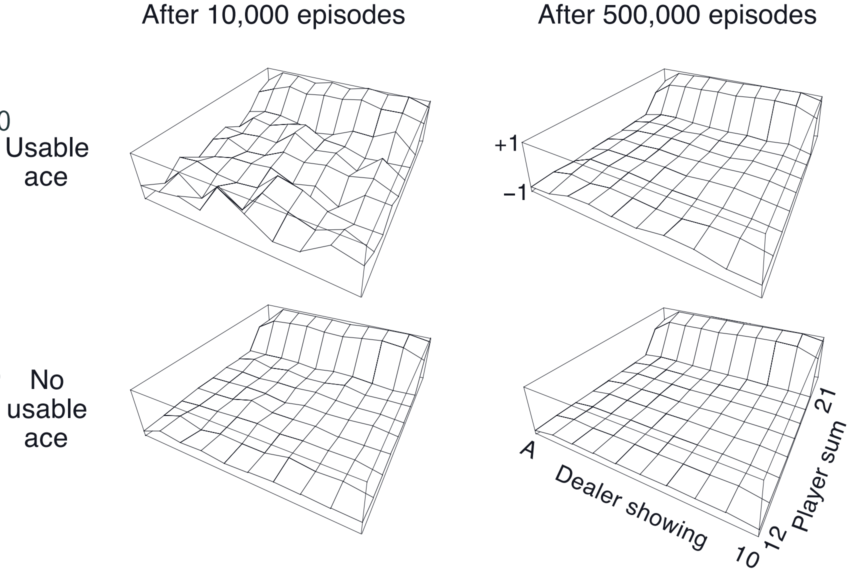

States in Blackjack

Couldn't we just define states as $S_t = \{\text{Player cards, Dealer card}\}$?

- Tricky: states would have variable length (player cards)
- If we fix maximum number of player cards to 4, then there are $10^5 = 100,000$ possible states! (ignoring face cards and ordering)

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Blackjack example uses engineered state features:

- Fixed length: $S_t = (\text{Player sum, Dealer card, Usable ace?})$
- Player sum limited to range 12–21 because decision below 12 is trivial (always hit)
- Number of states: $10 * 10 * 2 = 200 \rightarrow$ much smaller problem!
- Still has all relevant information

Blackjack and Dynamic Programming

Can we solve Blackjack MDP with DP methods?

- Yes, in principle, because we know complete MDP
- But computing $p(s', r|s, a)$ can be complicated!
  E.g. what is probability of $+1$ reward as function of Dealer's showing card?

Blackjack and Dynamic Programming

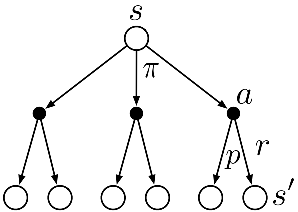
Can we solve Blackjack MDP with DP methods?

- Yes, in principle, because we know complete MDP
- But computing $p(s', r|s, a)$ can be complicated!
  E.g. what is probability of $+1$ reward as function of Dealer's showing card?
- On other hand, easy to code a simulation model:
  — Use Dealer rule to sample cards until stick/bust, then compute reward
  — Reward outcome is distributed by $p(s', r|s, a)$
- MC can evaluate policy without knowledge of probabilities $p(s', r|s, a)$

Monte Carlo Estimation of Action Values

MC methods can learn v_π without knowledge of model $p(s', r|s, a)$

$\Rightarrow$ But improving policy π from v_π requires model *(why?)*



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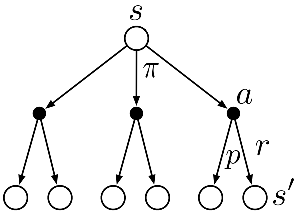


Must estimate action values:

$$q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$$

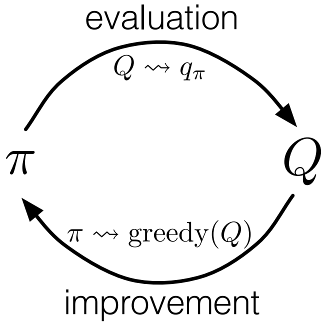
- Improve policy without model: $\pi'(s) = \arg\max_a q_\pi(s, a)$
- Use same MC methods to learn q_π, but visits are to (s, a)-pairs
- Converges to q_π if every (s, a)-pair visited infinitely many times in limit

  E.g. exploring starts: every (s, a)-pair has non-zero probability of being starting pair of episode

Monte Carlo Control

- **MC policy evaluation:**
  Estimate q_π using MC method

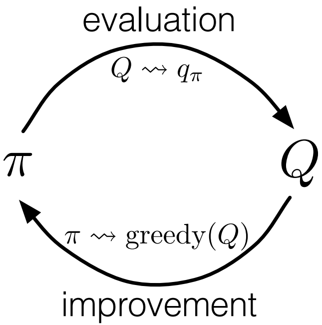
- **Policy improvement:**
  Improve π by making greedy wrt q_π



Monte Carlo Control with Exploring Starts

Greedy policy meets conditions for policy improvement theorem:

$$
\begin{aligned}
q_{\pi_k}(s, \pi_{k+1}(s)) &= q_{\pi_k}(s, \arg\max_a q_{\pi_k}(s, a)) \\
&= \max_a q_{\pi_k}(s, a) \\
&\geq q_{\pi_k}(s, \pi_k(s)) \quad (\textit{why?}) \\
&= v_{\pi_k}(s)
\end{aligned}
$$



Assumes exploring starts and *infinite* MC iterations

- In practice, we update only to a given performance threshold
- Or alternate between evaluation and improvement per episode

Monte Carlo Control with Exploring Starts

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:

$Q(s, a) \leftarrow$ arbitrary

$\pi(s) \leftarrow$ arbitrary

$Returns(s, a) \leftarrow$ empty list

Repeat forever:

Choose $S_0 \in \mathcal{S}$ and $A_0 \in \mathcal{A}(S_0)$ s.t. all pairs have probability > 0

Generate an episode starting from S_0, A_0, following π

For each pair s, a appearing in the episode:

$G \leftarrow$ return following the first occurrence of s, a

Append G to $Returns(s, a)$

$Q(s, a) \leftarrow \text{average}(Returns(s, a))$

For each s in the episode:

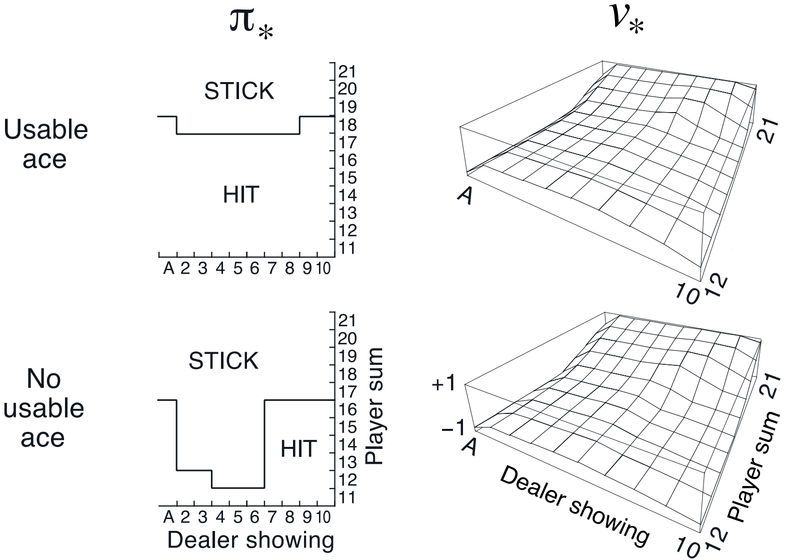
$\pi(s) \leftarrow \arg\max_a Q(s, a)$

Policy π:

stick if player sum is 20 or 21, else hit

Exploring starts:

sample initial states uniformly randomly



Monte Carlo Control with Soft Policies

Convergence to q_π requires that all (s, a)-pairs are visited infinitely many times

- Exploring starts guarantee this, but impractical *(why?)*

Monte Carlo Control with Soft Policies

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Other approach: use soft policy such that $\pi(a|s) > 0$ for all s, a

- e.g. ϵ-soft policy: $\pi(a|s) \geq \epsilon/|\mathcal{A}|$ for $\epsilon > 0$
- **Policy improvement:** make policy ϵ-greedy wrt q_π

$$\pi'(a|s) \doteq \begin{cases} \epsilon/|\mathcal{A}| + (1-\epsilon) & \text{if} \quad a = \arg\max_{a'} q_\pi(s, a') \\ \epsilon/|\mathcal{A}| & \text{else} \end{cases}$$

Monte Carlo Control with Soft Policies

ϵ-greedy policy meets conditions for policy improvement theorem:

$$
\begin{aligned}
q_\pi(s, \pi'(s)) &= \sum_a \pi'(a|s)\, q_\pi(s,a) \\
&= \frac{\epsilon}{|\mathcal{A}|} \sum_a q_\pi(s,a) + (1-\epsilon) \max_a q_\pi(s,a) \\
&\geq \frac{\epsilon}{|\mathcal{A}|} \sum_a q_\pi(s,a) + (1-\epsilon) \sum_a \frac{\pi(a|s) - \epsilon/|\mathcal{A}|}{1-\epsilon} q_\pi(s,a) \quad \textit{(why?)} \\
&= \frac{\epsilon}{|\mathcal{A}|} \sum_a q_\pi(s,a) - \frac{\epsilon}{|\mathcal{A}|} \sum_a q_\pi(s,a) + \sum_a \pi(a|s)\, q_\pi(s,a) \\
&= v_\pi(s)
\end{aligned}
$$

- Thus, π' better or equal to π, but both are still ϵ-soft
- $q_\pi(s, \pi'(s)) = v_\pi(s)$ only when π' and π both optimal ϵ-soft policies

Monte Carlo Control with Soft Policies

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:

$Q(s, a) \leftarrow$ arbitrary

$Returns(s, a) \leftarrow$ empty list

$\pi(a|s) \leftarrow$ an arbitrary ε-soft policy

Repeat forever:

(a) Generate an episode using π

(b) For each pair s, a appearing in the episode:

$G \leftarrow$ return following the first occurrence of s, a

Append G to $Returns(s, a)$

$Q(s, a) \leftarrow$ average$(Returns(s, a))$

(c) For each s in the episode:

$A^* \leftarrow \arg\max_a Q(s, a)$

For all $a \in \mathcal{A}(s)$:

$$\pi(a|s) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(s)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(s)| & \text{if } a \neq A^* \end{cases}$$

Off-Policy Methods

Like exploring starts, soft policies ensure all (s, a) are visited infinitely many times

- But policies restricted to be soft

  $\Rightarrow$ Optimal policy is usually deterministic!

- Could slowly reduce ϵ, but not clear how fast

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Other approach: off-policy learning

- Learn q_π based on experience generated with *behaviour policy* $\mu \neq \pi$
- Requires "coverage": if $\pi(a|s) > 0$ then $\mu(a|s) > 0$, for all s, a

  — e.g. use soft policy μ

- π can be deterministic $\rightarrow$ usually the greedy policy

Discussion: On-Policy vs Off-Policy Methods

On-policy:

Learn q_π with experience generated using policy π

Off-policy:

Learn q_π with experience generated using policy $\mu \neq \pi$

Importance Sampling Ratio

We have episodes generated from μ

$\Rightarrow$ Expected return at t is $\mathbb{E}_\mu[G_t|S_t = s] = v_\mu(s)$

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We have episodes generated from μ

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Fix expectation with sampling importance ratio:

$$\rho_{t:T} \doteq \frac{\prod_{k=t}^{T-1} \pi(A_k|S_k)\, p(S_{k+1}, R_{k+1}|S_k, A_k)}{\prod_{k=t}^{T-1} \mu(A_k|S_k)\, p(S_{k+1}, R_{k+1}|S_k, A_k)} = \prod_{k=t}^{T-1} \frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}$$

- $\mathbb{E}_\mu[\rho_{t:T}\, G_t|S_t = s] = v_\pi(s)$

Importance Sampling Ratio

$$\mathbb{E}_{\mu}[\rho_{t:T}\, G_t | S_t = s] = \sum_{E:S_t=s} \left[\prod_{k=t}^{T-1} \mu(A_k|S_k)\, p(S_{k+1}, R_{k+1}|S_k, A_k) \right] \rho_{t:T}\, G_t$$

$$= \sum_{E:S_t=s} \left[\prod_{k=t}^{T-1} \mu(A_k|S_k)\, p(S_{k+1}, R_{k+1}|S_k, A_k) \right] \prod_{k=t}^{T-1} \frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}\, G_t$$

$$= \sum_{E:S_t=s} \left[\prod_{k=t}^{T-1} \pi(A_k|S_k)\, p(S_{k+1}, R_{k+1}|S_k, A_k) \right] G_t$$

$$= v_{\pi}(s)$$

Evaluating Policies with Importance Sampling

Denote episodes $E^i = < S^i_0, A^i_0, R^i_1, S^i_1, A^i_1, R^i_2, ..., S^i_{T_i} >$

Define $\mathcal{E}(s)/\mathcal{E}(s,a)$ as before for first-visit or every-visit MC

Estimate v_π / q_π as

$$v_\pi(s) \;\approx\; \eta^{-1} \sum_{t_i \in \mathcal{E}(s)} \rho_{t_i:T_i}\, G^i_{t_i}$$

$$q_\pi(s,a) \;\approx\; \eta^{-1} \sum_{t_i \in \mathcal{E}(s,a)} \rho_{t_i+1:T_i}\, G^i_{t_i} \qquad \textit{(why } t_i + 1\textit{?)}$$

- **Ordinary** importance sampling: $\eta = |\mathcal{E}(s,a)|$
- **Weighted** importance sampling: $\eta = \sum_{t_i \in \mathcal{E}(s)} \rho_{t_i:T_i}$ resp. $\eta = \sum_{t_i \in \mathcal{E}(s,a)} \rho_{t_i+1:T_i}$

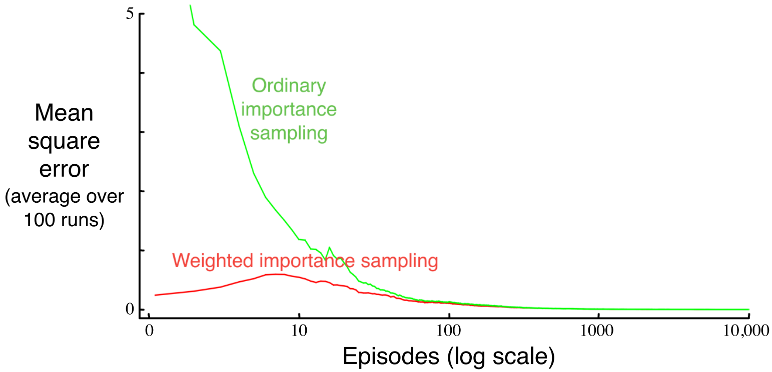
Off-Policy Value Estimation in Blackjack Example

π : stick if player sum is 20 or 21, else hit

μ : uniformly random

s : player sum 13
dealer showing 2
usable ace

True value:
$v_\pi(s) \approx -0.27726$



Reading

Required:

- RL book, Chapter 5 (5.1–5.7)

Optional:

- *Sequential Monte Carlo Methods in Practice*
  Arnaud Doucet, Nando de Freitas, Neil Gordon (editors)
  University library has copies