Reinforcement Learning

Temporal-Difference Learning

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Based on slides by Stefano V. Albrecht

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informatics

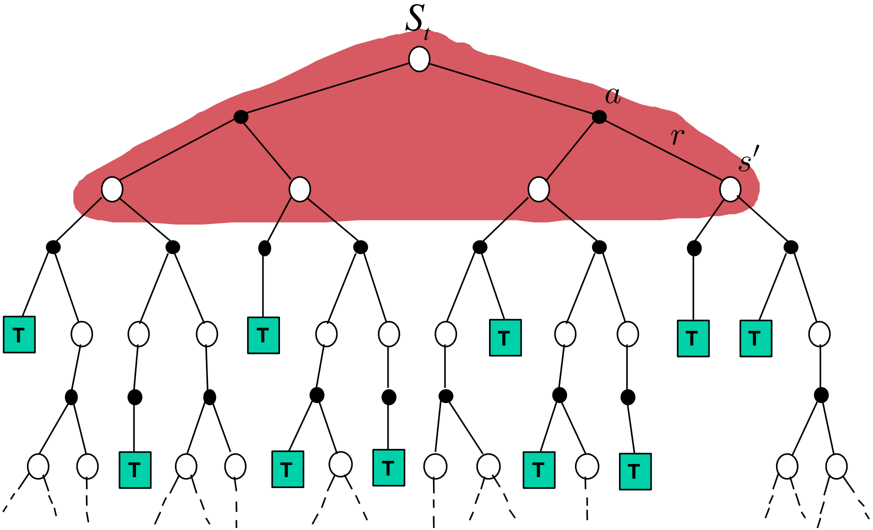
Lecture Outline

- Temporal-difference (TD) policy evaluation
- TD control:
  - Sarsa
  - Q-learning
- n-step TD methods

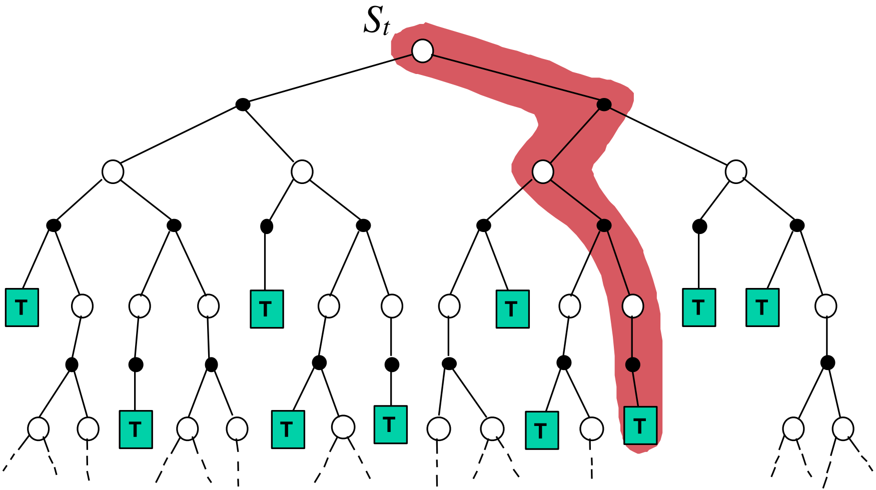
Method Comparison

Method	Model-free?	Bootstrap?
Dynamic Programming	No	Yes
Monte Carlo	Yes	No
Temporal-Difference	Yes	Yes

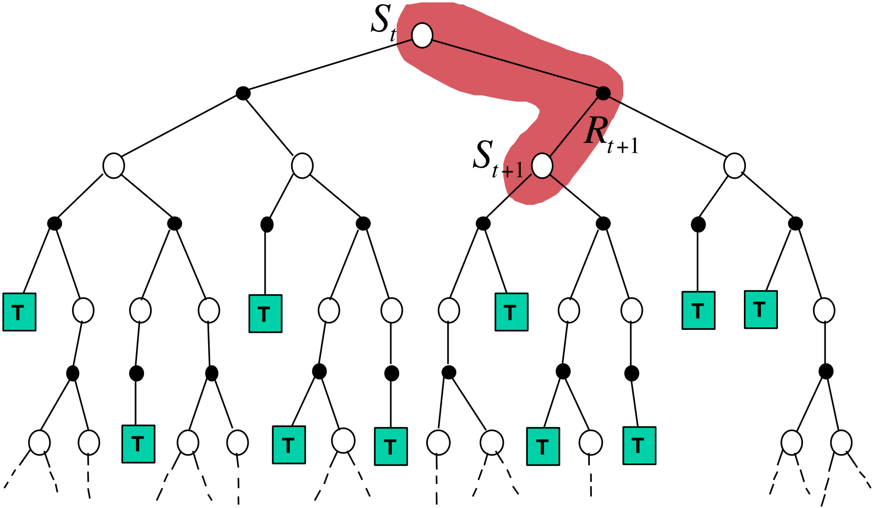
Recap: Dynamic Programming



Recap: Monte Carlo Methods



Now: Temporal-Difference Learning



Temporal-Difference Policy Evaluation

General iterative update rule:

$$\text{NewEstimate} \leftarrow \text{OldEstimate} + \text{StepSize}\,[\ \text{Target} - \text{OldEstimate}\ ]$$

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$$\text{NewEstimate} \leftarrow (1 - \underbrace{\text{StepSize}}_{\alpha})\ \text{OldEstimate} + \underbrace{\text{StepSize}}_{\alpha}\ \text{Target}$$

Temporal-Difference Policy Evaluation

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MC update:

$$V(S_t) \leftarrow V(S_t) + \alpha\,[\,G_t - V(S_t)\,]$$

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Notice:

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi[G_t | S_t = s] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} | S_t = s] \\ &= \mathbb{E}_\pi[\underbrace{R_{t+1} + \gamma v_\pi(S_{t+1})}_{\text{Use as target}} | S_t = s] \end{aligned}$$

Temporal-Difference Policy Evaluation

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MC update:

$$V(S_t) \leftarrow V(S_t) + \alpha\,[\ G_t - V(S_t)\ ]$$

TD(0) update:

$$V(S_t) \leftarrow V(S_t) + \alpha\,\underbrace{[\ R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\,]}_{\text{“}\delta-\text{error”}}$$

TD(0) for Policy Evaluation

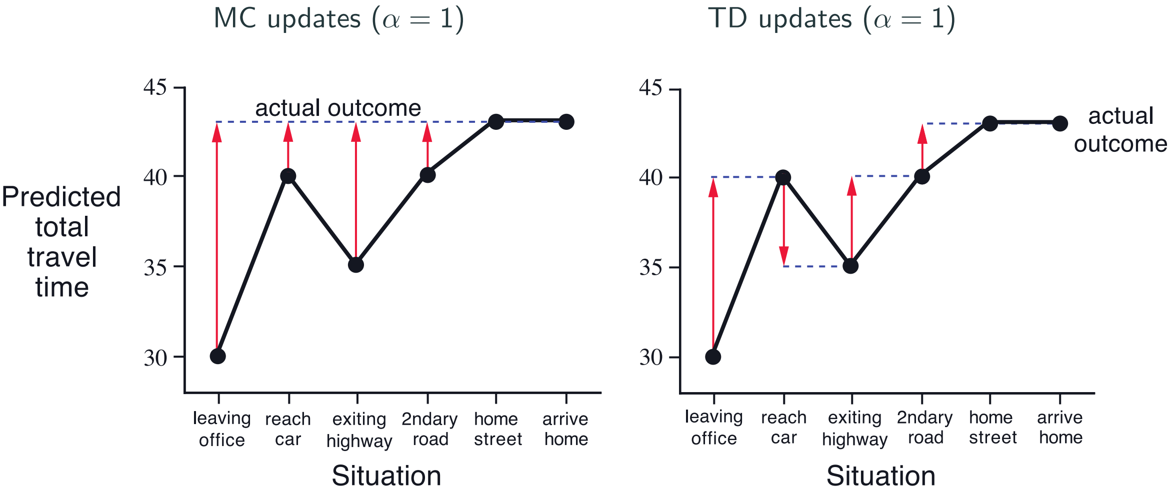
Input: the policy π to be evaluated
Algorithm parameter: step size $\alpha \in (0, 1]$
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(\mathit{terminal}) = 0$

Loop for each episode:
  Initialize S
  Loop for each step of episode:
    $A \leftarrow$ action given by π for S
    Take action A, observe R, S'
    $V(S) \leftarrow V(S) + \alpha\big[R + \gamma V(S') - V(S)\big]$
    $S \leftarrow S'$
  until S is terminal

Example: Driving Home

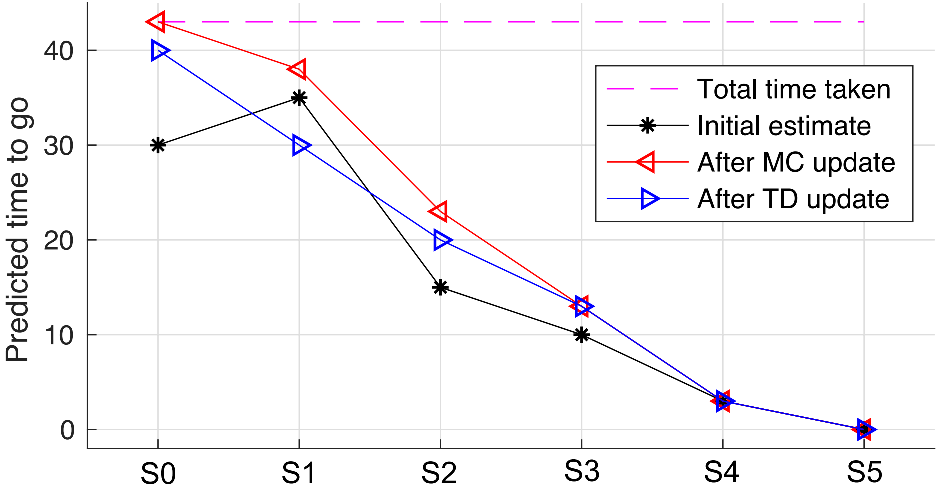
	$(\gamma = 1)$	$\sum_{k=1}^{t} R_k$	$V(S_t)$	$V(S_0)$
	State	*Elapsed Time (minutes)*	*Predicted Time to Go*	*Predicted Total Time*
S_0	leaving office, friday at 6	0	30	30
S_1	reach car, raining	5	35	40
S_2	exiting highway	20	15	35
S_3	2ndary road, behind truck	30	10	40
S_4	entering home street	40	3	43
S_5	arrive home	43	0	43

Example: Driving Home



Example: Driving Home (Extra)

$V(S_t)$



Convergence of TD(0)

TD(0) converges to v_π with prob. 1 if:

- all states visited infinitely often

  and

- standard stochastic approximation conditions (α-reduction)

$$\forall s : \quad \sum_{t:S_t=s} \alpha_t \to \infty \quad \text{and} \quad \sum_{t:S_t=s} \alpha_t^2 < \infty$$

Convergence of TD(0)

Intuition: What is the *expected* TD(0) update?

$$
\begin{aligned}
V(S_t) &\leftarrow \mathbb{E}_\pi[(1-\alpha)V(S_t) + \alpha\,[R_{t+1} + \gamma V(S_{t+1})]] && \text{(rewrite)} \\
&= (1-\alpha)V(S_t) + \alpha\mathbb{E}_\pi[R_{t+1} + \gamma V(S_{t+1})] \\
&= (1-\alpha)V(S_t) + \alpha \sum_a \pi(a|S_t) \sum_{s',r} p(s', r|S_t, a)\left[r + \gamma V(s')\right] \\
&= (1-\alpha)V(S_t) + \alpha\, v_\pi(S_t)
\end{aligned}
$$

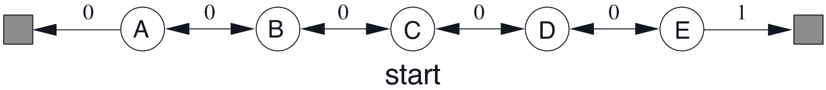
Bellman operator $v_\pi(S_t)$ is contraction mapping with fixed point v_π!

- Expected TD update moves $V(S_t)$ toward $v_\pi(S_t)$ by α
- α used to control averaging in sampling updates

Advantages of TD Learning

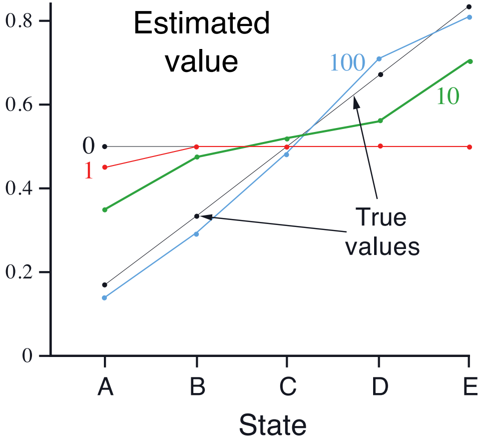
- Like MC: TD does not require full model $p(s', r|s, a)$, only *experience*
- Unlike MC: TD can be fully *incremental*
  - $\Rightarrow$ Learn *before* final return is known
  - $\Rightarrow$ Less memory and computation
- Both MC and TD converge to v_π / q_π under certain assumptions
  - $\Rightarrow$ But TD often faster in practice

Example: Random Walk

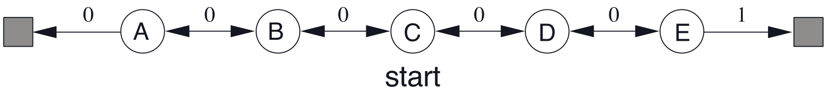


$\pi(left/right|s) = 0.5$

Values learned by TD(0) after 0/1/10/100 episodes ($\alpha = 0.1$)



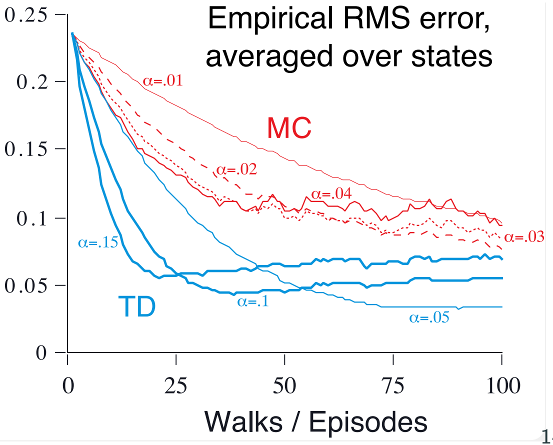
Example: Random Walk



$\pi(left/right|s) = 0.5$

Root mean-squared error averaged over all states and 100 episodes

TD methods usually learn faster than MC



On-Policy TD Control: Sarsa

On-policy: learn q_π and improve π while following π

Sarsa:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left[\, R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t) \, \right]$$

- If S_{t+1} terminal state, define $Q(S_{t+1}, A_{t+1}) = 0$
- Ensure exploration by using ϵ-soft policy π

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- Ensure exploration by using ϵ-soft policy π

Converges to π_* with prob 1. if all (s, a) infinitely visited and standard α-reduction

$$\forall s, a: \quad \sum_{t: S_t = s, A_t = a} \alpha_t \to \infty, \quad \sum_{t: S_t = s, A_t = a} \alpha_t^2 < \infty$$

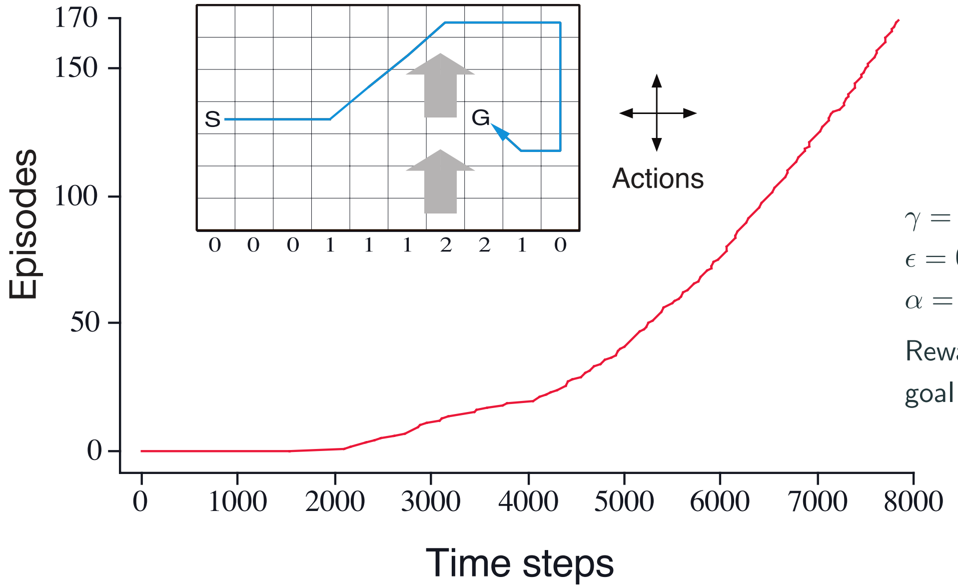
and ϵ gradually goes to 0 *(why?)*

On-Policy TD Control: Sarsa

See Tutorial 5

Initialize $Q(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
    Initialize S
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Repeat (for each step of episode):
        Take action A, observe R, S'
        Choose A' from S' using policy derived from Q (e.g., ε-greedy)
        $Q(S, A) \leftarrow Q(S, A) + \alpha[R + \gamma Q(S', A') - Q(S, A)]$
        $S \leftarrow S'$; $A \leftarrow A'$;
    until S is terminal

Example: Windy Gridworld with Sarsa



$\gamma = 1.0$

$\epsilon = 0.1$

$\alpha = 0.5$

Reward -1 until goal reached

Off-Policy TD Control: Q-Learning

Off-policy: Learn q_π and improve π while following μ

Q-learning:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t) \right]$$

Converges to π_* with prob. 1 if all (s, a) infinitely visited and standard α-reduction

Off-Policy TD Control: Q-Learning

Off-policy: Learn q_π and improve π while following μ

Q-learning:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t) \right]$$

Converges to π_* with prob. 1 if all (s, a) infinitely visited and standard α-reduction

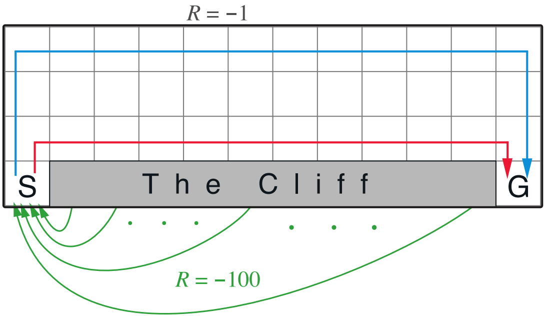
Why is there no importance sampling ratio?

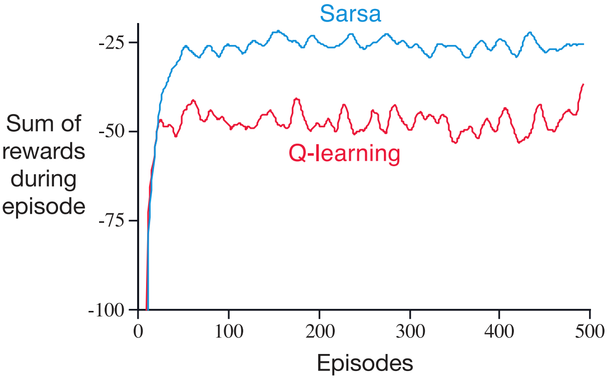
- Recall: for q_π, ratio defined as $\prod_{k=t+1}^{T-1} \pi(A_k|S_k)/\mu(A_k|S_k)$
- Because a in $q_\pi(s, a)$ is no random variable

Off-Policy TD Control: Q-Learning

Initialize $Q(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
    Initialize S
    Repeat (for each step of episode):
        Choose A from S using policy derived from Q (e.g., ε-greedy)
        Take action A, observe R, S'
        $Q(S, A) \leftarrow Q(S, A) + \alpha[R + \gamma \max_a Q(S', a) - Q(S, A)]$
        $S \leftarrow S'$;
    until S is terminal

Example: Cliff Walking with Sarsa and Q-Learning





ϵ-greedy exploration ($\epsilon = 0.1$)

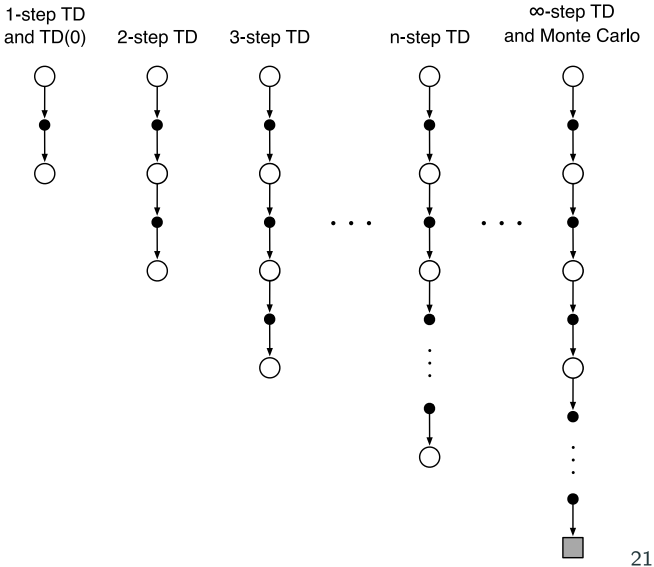
n-step TD Methods

TD(0) uses 1-step return:

$$G_{t:t+1} \doteq R_{t+1} + \gamma V_t(S_{t+1})$$

MC uses full return:

$$G_{t:\infty} \doteq \sum_{k=1}^{\infty} \gamma^{k-1} R_{t+k}$$



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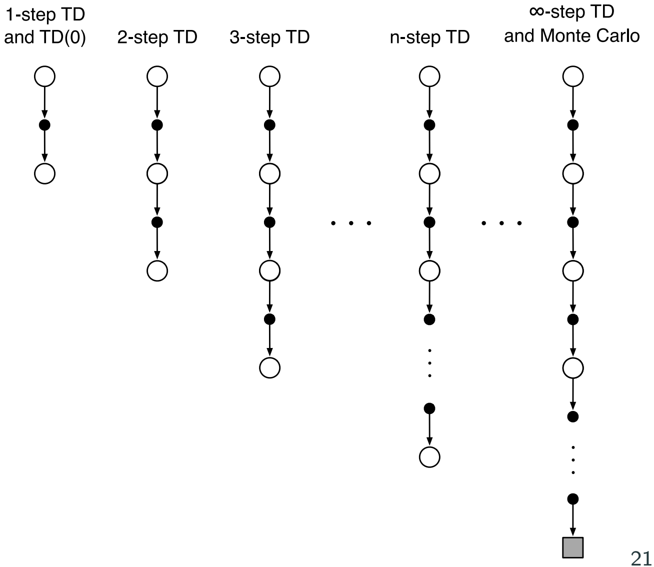
n-step return bridges TD(0) and MC:

$$G_{t:t+n} = \sum_{k=1}^{n} \gamma^{k-1} R_{t+k} \; + \; \gamma^n V_{t+n-1}(S_{t+}$$



n-step TD Methods

n-step return:

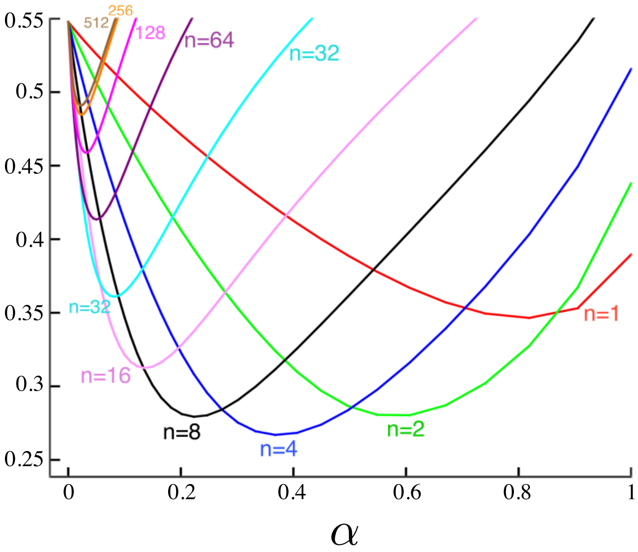
$$G_{t:t+n} = \sum_{k=1}^{n} \gamma^{k-1} R_{t+k} + \gamma^{n} V_{t+n-1}(S_{t+n})$$

n-step TD uses n-step return as target:

$$V_{t+n}(S_t) \doteq V_{t+n-1}(S_t) + \alpha \left[G_{t:t+n} - V_{t+n-1}(S_t) \right]$$

n-step TD Methods in Random Walk Example

Average RMS error over 19 states and first 10 episodes



On/Off-Policy Learning with n-Step Returns

Can similarly define *n*-step TD policy learning:

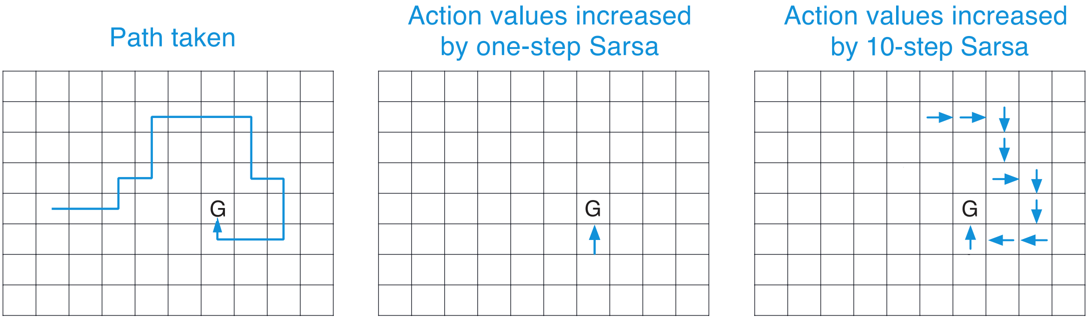
$$G_{t:t+n} = \sum_{k=1}^{n} \gamma^{k-1} R_{t+k} \; + \; \gamma^n Q_{t+n-1}(S_{t+n}, A_{t+n})$$

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha \rho_{t+1:t+n} \left[G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \right]$$

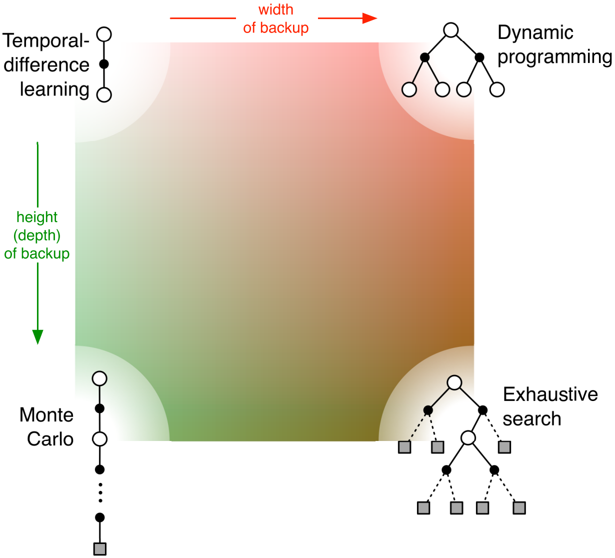
with importance ratio

$$\rho_{t:h} \doteq \prod_{k=t}^{\min(h, T-1)} \frac{\pi(A_k | S_k)}{\mu(A_k | S_k)}$$

n-step TD Control in a Gridworld



Unified View



$\Rightarrow$ Planning

Reading

Required:

- RL book, Chapter 6 (6.1–6.2, 6.4–6.6) and Chapter 7 (7.1–7.3)

Optional (convergence proofs):

- **For TD(0):** Dayan, P. (1992). The convergence of TD(λ) for general λ. Machine Learning, 8(3):341–362
- **For Sarsa:** Singh, S., Jaakkola, T., Littman, M., Szepesvári, C. (2000). Convergence results for single-step on-policy reinforcement-learning algorithms. Machine Learning, 38(3):287–308
- **For Q-learning:** Watkins, C., Dayan, P. (1992). Q-learning. Machine Learning, 8(3-4):279–292