Automated Reasoning

Lecture 4: Propositional Reasoning in Isabelle

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Recap

Last lecture:

- Completed the natural deduction system for propositional logic
- Started on proving propositions in Isabelle

Today:

- More details on proving propositions in Isabelle
- Alternative inference rules (L-system, a.k.a. “Sequent Calculus”)
- Why should we trust Isabelle?

The `rule` Method

To apply an inference rule, we use `rule`.

Consider the theorem `disjI1`

$$?P \Longrightarrow ?P \vee ?Q$$

Using the command

```
apply (rule disjI1)
```

on the goal

$$[\![A; B; C]\!] \Longrightarrow (A \wedge B) \vee D$$

yields the subgoal

$$[\![A; B; C]\!] \Longrightarrow A \wedge B$$

General definition of method `rule`

When we apply the method `rule someRule` where

$$\texttt{someRule} : [\![P_1; \ldots; P_m]\!] \Longrightarrow Q$$

to the goal

$$[\![A_1; \ldots; A_n]\!] \Longrightarrow C$$

where Q and C can be *unified*, we generate the goals

$$\begin{array}{c} [\![A'_1; \quad \ldots; \quad A'_n]\!] \Longrightarrow P'_1 \\ \vdots \\ [\![A'_1; \quad \ldots; \quad A'_n]\!] \Longrightarrow P'_m \end{array}$$

where $A'_1, A'_2, \ldots, A'_n, P'_1, P'_2, \ldots, P'_m$ are the results of applying the substitution which unifies Q and C to $A_1, A_2, \ldots, A_n, P_1, P_2, \ldots, P_m$.

We must now derive each of the rule's assumptions using our goal's assumptions.

A Problem with `rule`

Consider the `disjE` rule:

$$\texttt{disjE} : [\![?P \vee ?Q; ?P \Longrightarrow ?R; ?Q \Longrightarrow ?R]\!] \Longrightarrow ?R$$

If we have the goal:

$$[\![(A \wedge B) \vee C; D]\!] \Longrightarrow B \vee C$$

Then applying `rule disjE` produces three new goals:

$$[\![(A \wedge B) \vee C; D]\!] \Longrightarrow ?P \vee ?Q$$
$$[\![(A \wedge B) \vee C; D; ?P]\!] \Longrightarrow B \vee C$$
$$[\![(A \wedge B) \vee C; D; ?Q]\!] \Longrightarrow B \vee C$$

We then solve the first subgoal by applying `assumption`.

This seems pointlessly roundabout… we often want to *use* one of our assumptions in our proof.

The `erule` Method

Used when the conclusion of theorem matches the conclusion of the current goal **and** the first premise of theorem matches a premise of the current goal.

Consider the theorem `disjE` [1]

$$[\![P \vee Q; P \Longrightarrow R; Q \Longrightarrow R]\!] \Longrightarrow R$$

Applying `erule disjE` to goal

$$[\![(A \wedge B) \vee C; D]\!] \Longrightarrow B \vee C$$

yields the subgoals

$$[\![D; (A \wedge B)]\!] \Longrightarrow B \vee C \qquad [\![D; C]\!] \Longrightarrow B \vee C$$

[1] with the ? in front of variables omitted

General definition of method `erule`

When we apply the method `erule someRule` where

$$\texttt{someRule} : [\![P_1; \ldots; P_m]\!] \Longrightarrow Q$$

to the goal

$$[\![A_1; \ldots; A_n]\!] \Longrightarrow C$$

where P_1 and A_1 are unifiable and Q and C are unifiable, we generate the goals:

$$\begin{array}{llll} [\![A_2'; & \ldots; & A_n']\!] \Longrightarrow P_2' \\ & \vdots & \\ [\![A_2'; & \ldots; & A_n']\!] \Longrightarrow P_m' \end{array}$$

where $A_2', \ldots, A_n', P_2', \ldots, P_m'$ are the results of applying the substitution which unifies P_1 to A_1 and Q to C to $A_2, \ldots, A_n, P_2, \ldots, P_m$.

We **eliminate** an assumption from the rule and the goal, and must derive the rule's other assumptions using our goal's other assumptions.

General definition of method `drule`

When we apply the method `drule someRule` where

$$\texttt{someRule} : [\![P_1; \ldots; P_m]\!] \Longrightarrow Q$$

to the goal

$$[\![A_1; \ldots; A_n]\!] \Longrightarrow C$$

where P_1 and A_1 are unifiable, we generate the goals:

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where $A_2', A_3', \ldots, A_n', P_2', P_3' \ldots, P_m', Q', C'$ are the results of applying the substitution which unifies P_1 and A_1 to $A_2, A_3, \ldots, A_n, P_2, P_3, \ldots, P_m, Q, C$.

We **delete** an assumption, replacing it with the conclusion of the rule.

General definition of method `frule`

When we apply the method `frule someRule` where

$$\texttt{someRule} : [\![P_1; \ldots; P_m]\!] \Longrightarrow Q$$

to the goal

$$[\![A_1; \ldots; A_n]\!] \Longrightarrow C$$

where P_1 and A_1 are unifiable, we generate the goals:

$$\begin{array}{rcl}
[\![A_1'; A_2'; & \ldots; & A_n']\!] \Longrightarrow P_2' \\
& \vdots & \\
[\![A_1'; A_2'; & \ldots; & A_n']\!] \Longrightarrow P_m' \\
[\![Q'; A_1'; A_2'; & \ldots; & A_n']\!] \Longrightarrow C'
\end{array}$$

where $A_1', A_2', \ldots, A_n', P_2', \ldots, P_m', Q', C'$ are the results of applying the substitution which unifies P_1 and A_1 to $A_1, A_2, \ldots, A_n, P_2, \ldots, P_m, Q, C$.

This is like `drule` except the assumption in our goal is kept.

More Methods

- `rule_tac`, `erule_tac`, `drule_tac` and `frule_tac` are like their counterparts, but you can give substitutions for variables in the rule before they are applied.

Example

```
apply (erule_tac Q="B ∧ D" in conjE)
```

applied to the subgoal

$$[\![A \wedge B; C \wedge B \wedge D]\!] \Longrightarrow B \wedge D$$

generates the new goal

$$[\![A \wedge B; C; B \wedge D]\!] \Longrightarrow B \wedge D$$

- conjE: $[\![P \wedge Q; [\![P; Q]\!] \Longrightarrow R]\!] \Longrightarrow R$
- Isabelle also provides advanced tactics, like `simp` and `auto` which perform some **automatic deduction**.

L-systems/Sequent Calculus

The `erule` tactic points to another way of phrasing a system of inference rules in a system with sequents $\Gamma \vdash A$.

Instead of *elimination* rules:

$$\frac{\Gamma \vdash P \vee Q \qquad \Gamma, P \vdash R \qquad \Gamma, Q \vdash R}{\Gamma \vdash R} \text{ (disjE)}$$

Have *left introduction rules* (all the introduction rules in natural deduction introduce connectives on the right-hand side of the $\vdash$):

$$\frac{\Gamma, P \vdash R \qquad \Gamma, Q \vdash R}{\Gamma, P \vee Q \vdash R}$$

This corresponds to applying rules using `erule` in Isabelle.

The *left introduction rules* are often much easier to use in a backwards, goal-directed style.

L-systems/Sequent Calculus

The following *L*-System (a.k.a. Sequent Calculus) rules are an alternative sound and complete proof system for propositional logic:

$$\frac{}{\Gamma, P \vdash P}\ \text{(assumption)}$$

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ \text{(conjI)} \qquad \frac{\Gamma, P, Q \vdash R}{\Gamma, P \wedge Q \vdash R}\ \text{(e conjE)}$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\ \text{(disjI1)} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\ \text{(disjI2)} \qquad \frac{\Gamma, P \vdash R \qquad \Gamma, Q \vdash R}{\Gamma, P \vee Q \vdash R}\ \text{(e disjE)}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ \text{(impI)} \qquad \frac{\Gamma \vdash P \qquad \Gamma, Q \vdash R}{\Gamma, P \rightarrow Q \vdash R}\ \text{(e impE)}$$

no right-intro rule for $\bot$ $\qquad \frac{}{\Gamma, \bot \vdash P}\ \text{(e FalseE)}$

$$\frac{\Gamma, P \vdash \bot}{\Gamma \vdash \neg P}\ \text{(notI)} \qquad \frac{\Gamma \vdash P}{\Gamma, \neg P \vdash R}\ \text{(e notE)} \qquad \frac{}{\Gamma \vdash \neg P \vee P}\ \text{(excluded\_middle)}$$

Note: `e someRule` is short for `erule someRule`.

Note: in the above presentation left-hand-sides are *sets* of formulas.

An Old Friend Revisited

$$\dfrac{\dfrac{\dfrac{\dfrac{}{S \vdash S}\,(\text{assumption})}{S, \neg S \vdash R}\,(\text{e notE}) \qquad \dfrac{}{R, \neg S \vdash R}\,(\text{assumption})}{(S \vee R), \neg S \vdash R}\,(\text{e disjE})}{(S \vee R) \wedge \neg S \vdash R}\,(\text{e ConjE})}{\vdash (S \vee R) \wedge \neg S \rightarrow R}\,(\text{impI})$$

Re-using proofs: The Cut rule

So far, all proofs have been self-contained; they have only used the pre-existing rules of inference.

By the completeness theorem, this suffices to prove everything that is true, but can lead to extremely repetitive proofs.

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$$\frac{\Gamma \vdash P \qquad \Gamma, P \vdash Q}{\Gamma \vdash Q}$$

allows the use of a *lemma* P in a proof of Q. We can now reuse P multiple times in the proof of Q.

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In Isabelle:

`cut_tac lemmaName`	— adds the conclusion of `lemmaName` as a new assumption, and its assumptions as new subgoals
`subgoal_tac` P	— adds P as a new assumption, and introduces P as a new subgoal.

Why should you believe Isabelle?

When Isabelle says "`No subgoals!`" why should we believe that we have *really* proved something? Is Isabelle sound?

It is doing non-trivial work behind the scenes: unification, rewriting, maintaining a database of theorems+assumptions, automatic proof.

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Isabelle uses two strategies to maintain soundness:

- A small trusted kernel: internally, every proof is broken down into primitive rule applications which are checked by a small piece of hand-verified code. This is the "LCF" model. So new *proof procedures* cannot introduce unsoundness.
- Encourages *definitional* extension of the logic: new concepts are introduced by definition rather than axiomatisation (more on this in Lecture 6). So new definitions cannot introduce unsoundness.

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Threats to (practical) soundness still exist, including: Have we proved what we thought we proved? Are the formulas displayed on screen correctly? ...

See: Pollack, R. *How to Believe a Machine-Checked Proof*, 1997 (non-examinable).

Summary

- More tools for proving propositions in Isabelle
  - The `erule`, `drule`, `frule` methods
  - Their `_tac` variants
  - *L*-systems, and Cut rules (`cut_tac`, `subgoal_tac`).
  - See the propositional logic exercises and examples:
    - Self-help Exercise Sheet 1 and Additional Exercise on the AR webpage;
    - The Isabelle theory file `Prop.thy`;
    - Start using Isabelle (if you haven't done so already).
- How Isabelle maintains soundness
  - Small trusted kernel
  - Definitional extension instead of axiomatic extension
- Next time:
  - First-Order Logic: $\forall x.P$ and $\exists x.P$